# Three-body problem in GR

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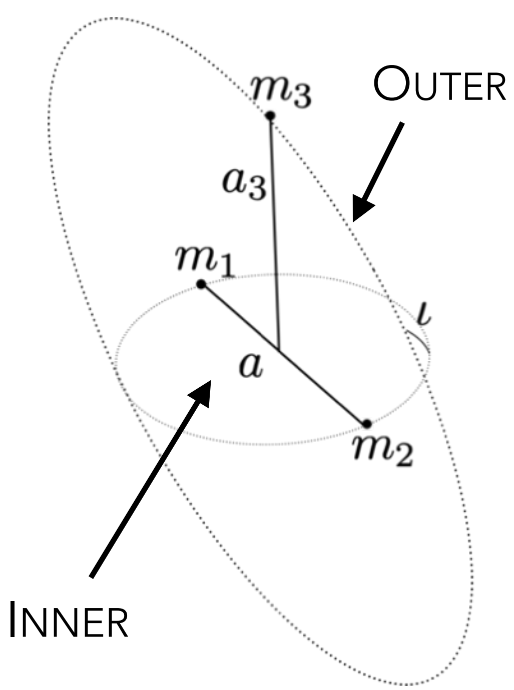
In collaboration with : Enrico Trincherini, Francesco Serra, Konstantin Leyde

# INTRODUCTION

## SOME HISTORY OF THE THREE-BODY PROBLEM

- Until end of 18th century, it was still unclear if Newton's law could explain the orbits of Solar System

- Lagrange and Laplace inaugurated methods of celestial mechanics

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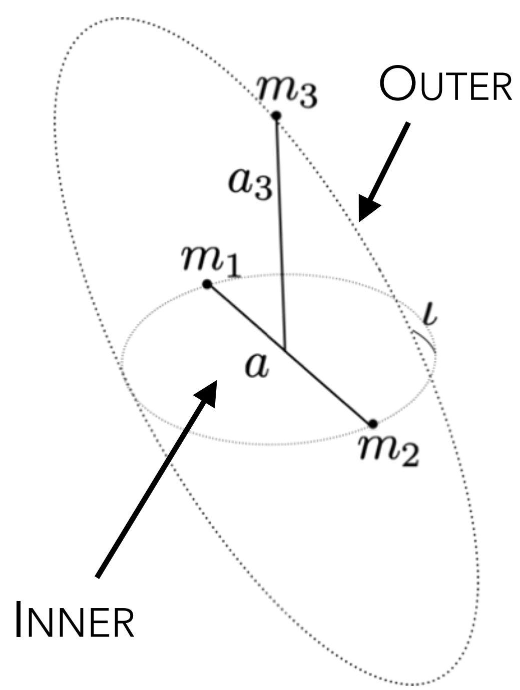


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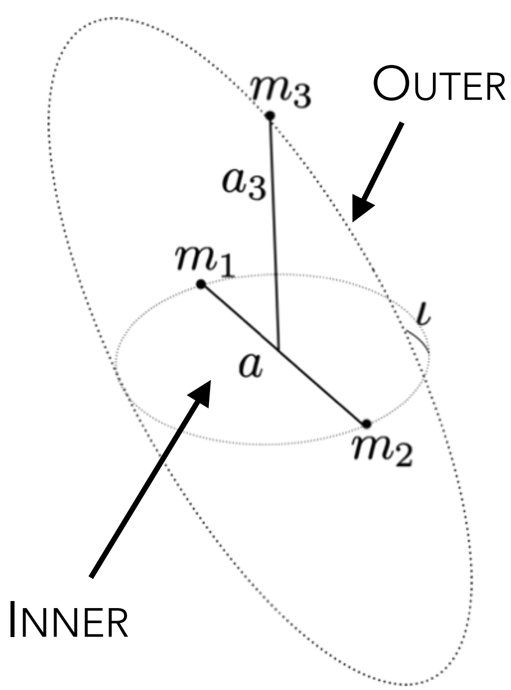


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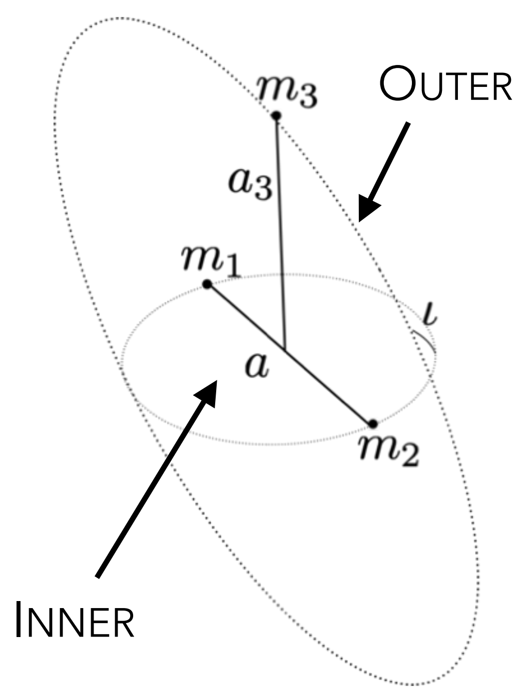


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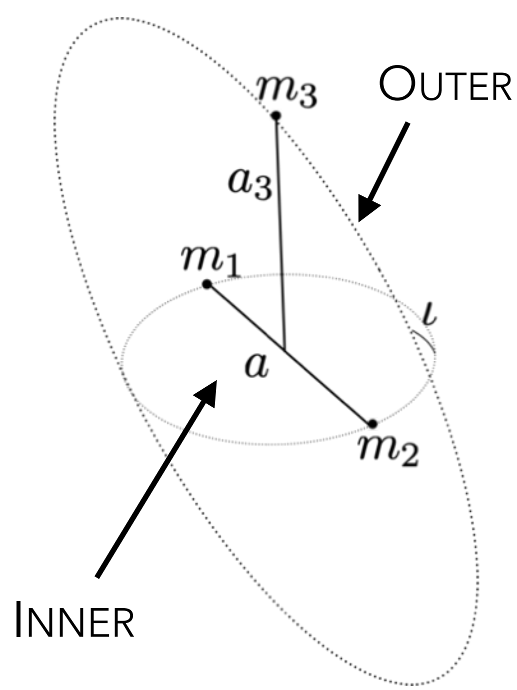


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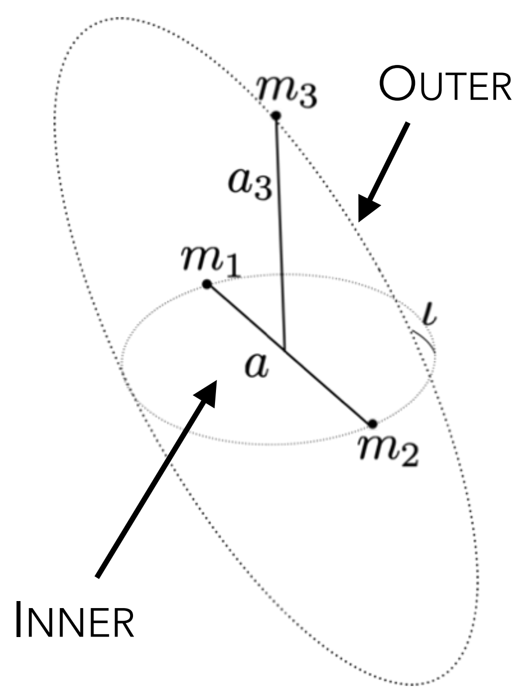


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$$\mathcal{H} = \mathcal{H}_{\mathrm{inner}} + \mathcal{H}_{\mathrm{outer}} + \mathcal{H}_{\mathrm{int}}$$

$$\mathcal{H}_{\mathrm{int}} = \frac{G m_1 m_2 m_3}{2 m a_3} \left( \frac{a}{a_3} \right)^2 (3 \cos^2 \psi - 1)$$

# INTRODUCTION

## A SMALL CLOUD IN A BLUE SKY

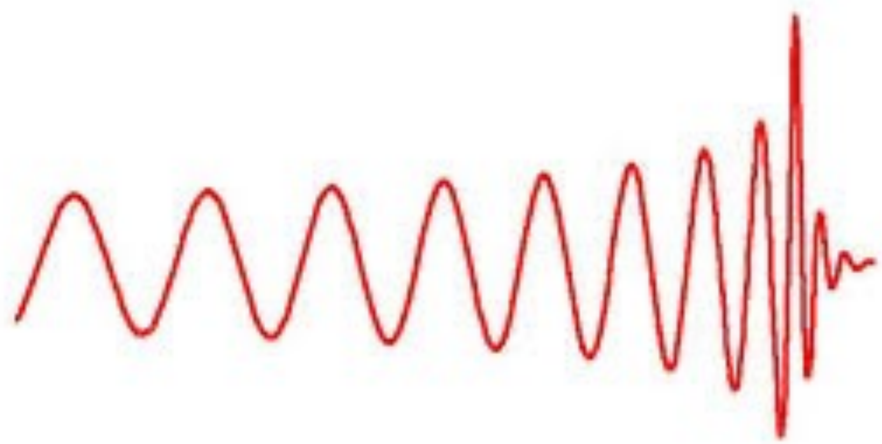
- Careful analysis showed supplementary precession for Mercury…

| **Amount** ('' $cy^{-1}$) | **Cause** |
|---|---|
| $531.63 \pm 0.69$ | gravitational tugs from the other planets |
| 0.025 4 | oblateness of the Sun |
| $42.98 \pm 0.04$ | general relativity |
| $574.64 \pm 0.69$ | total |
| $574.10 \pm 0.65$ | observed |

# GRAVITATIONAL WAVES

## A BIG JUMP IN HISTORY

Detection of GW so far beautifully corresponds to two-body systems



$$\Phi(f) = \phi_0 + 2\pi f t_0 + \sum_{k=0}^{7} \alpha_k f^{(k-5)/3}$$

$$m_1, m_2, \chi_1, \chi_2$$

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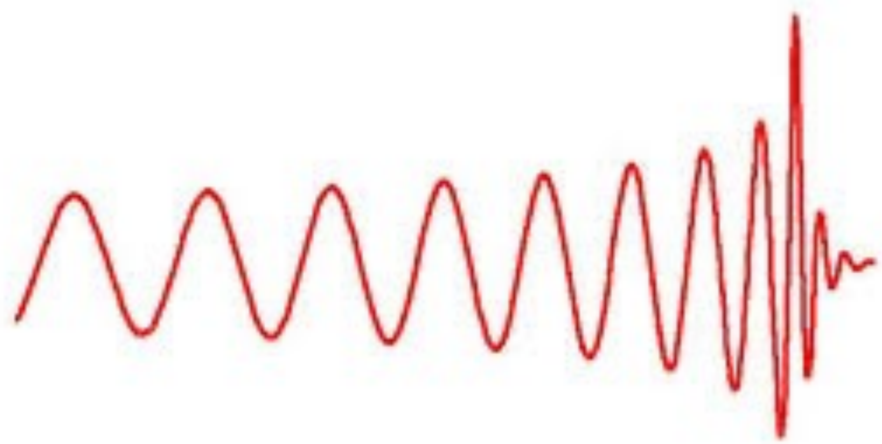

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If we ever detect a new feature in data, we have (as 19th century astronomers) two possible explanations:

- Modification of GR
- Perturbation by a third body (this talk)

# GRAVITATIONAL WAVES

## THIS QUESTION IS NOT PURELY ACADEMIC !

Three-body systems are also quite common !

- 90% of low-mass binaries are expect to belong to a 'hierarchical' triple system

Tokovinin et al. 2006

- 'Migration traps' around SMBH at $R \sim 20 - 600 R_{\rm sch}$

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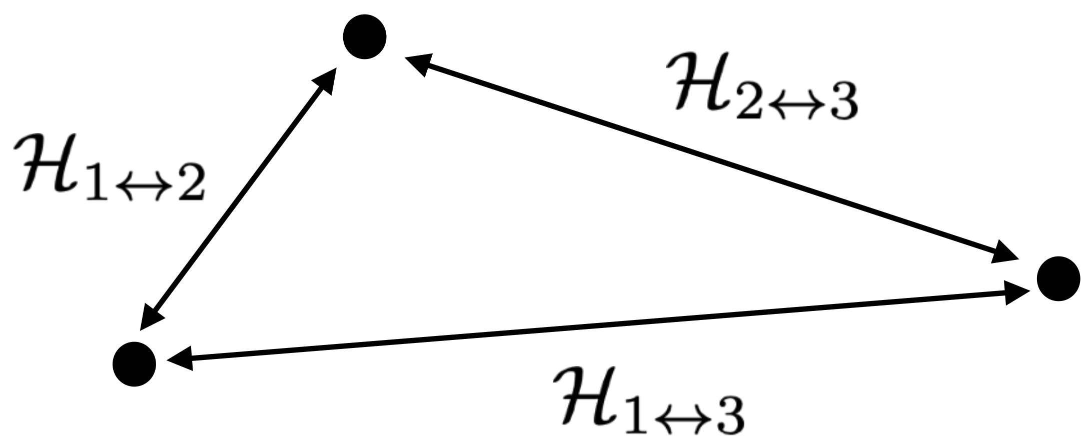
Can we detect and measure parameters of the third body from waveform ?

$\Rightarrow$ As Lagrange and Laplace, we have to formulate the 3-body problem in GR and solve it perturbatively

# Relativistic three-body problem

# A COMMON MISCONCEPTION

$$\mathcal{H}_{3-\mathrm{body}} = \mathcal{H}_{1\leftrightarrow 2} + \mathcal{H}_{1\leftrightarrow 3} + \mathcal{H}_{2\leftrightarrow 3}$$



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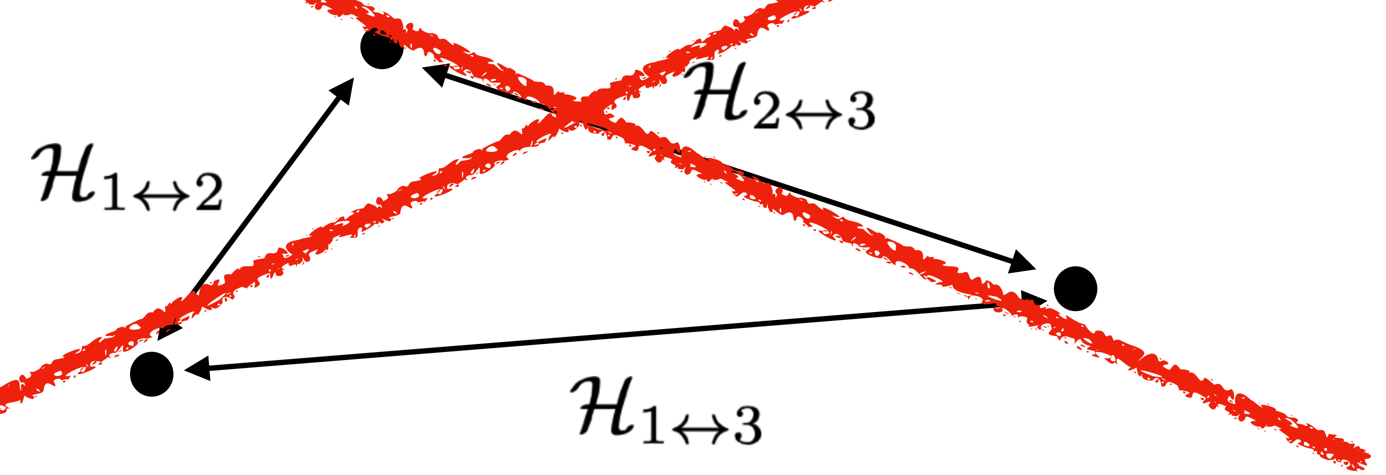
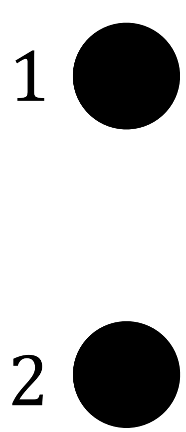



GR IS A NONLINEAR THEORY !




$$g_{\mu\nu} \neq g^{(1)}_{\mu\nu} + g^{(2)}_{\mu\nu} \quad \text{(would not solve } R_{\mu\nu} = 0\text{ )}$$

# GR CORRECTIONS AT 1PN

EOM in post-Newtonian

$$\boldsymbol{a}_a = -\sum_{b\neq a} \frac{Gm_b \boldsymbol{x}_{ab}}{r_{ab}^3} + \frac{1}{c^2} \sum_{b\neq a} \frac{Gm_b \boldsymbol{x}_{ab}}{r_{ab}^3} \left[ 4\frac{Gm_b}{r_{ab}} + 5\frac{Gm_a}{r_{ab}} + \sum_{c\neq a,b} \frac{Gm_c}{r_{bc}} + 4\sum_{c\neq a,b} \frac{Gm_c}{r_{ac}} - \frac{1}{2}\sum_{c\neq a,b} \frac{Gm_c}{r_{bc}^3}(\boldsymbol{x}_{ab}\cdot\boldsymbol{x}_{bc}) - v_a^2 + 4\boldsymbol{v}_a\cdot\boldsymbol{v}_b \right.$$
$$\left. - 2\boldsymbol{v}_b^2 + \frac{3}{2}(\boldsymbol{v}_b\cdot\boldsymbol{n}_{ab})^2 \right] - \frac{7}{2c^2}\sum_{b\neq a}\frac{Gm_b}{r_{ab}}\sum_{c\neq a,b}\frac{Gm_c \boldsymbol{x}_{bc}}{r_{bc}^3} + \frac{1}{c^2}\sum_{b\neq a}\frac{Gm_b}{r_{ab}^3}\boldsymbol{x}_{ab}\cdot(4\boldsymbol{v}_a - 3\boldsymbol{v}_b)(\boldsymbol{v}_a - \boldsymbol{v}_b), \tag{3.1}$$

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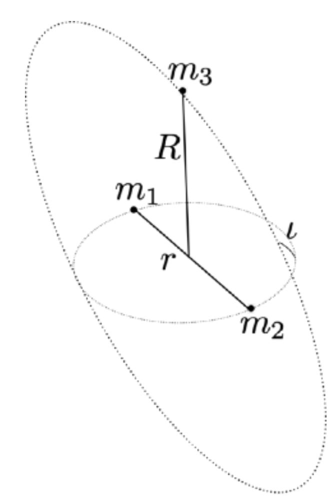


- Lack of physical intuition

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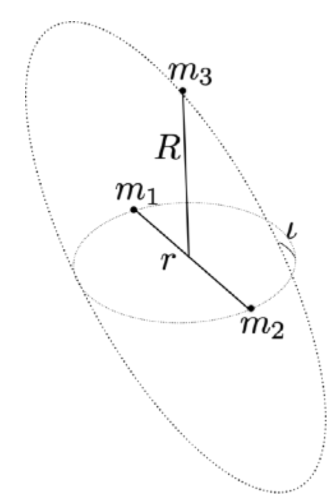


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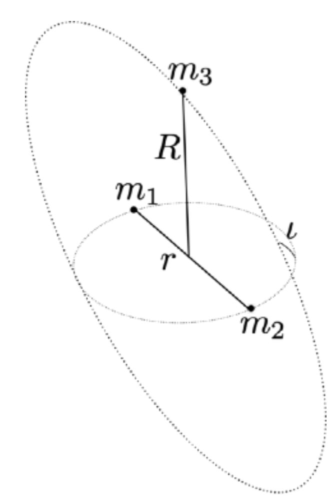
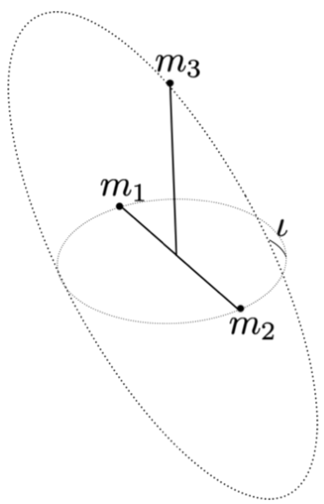


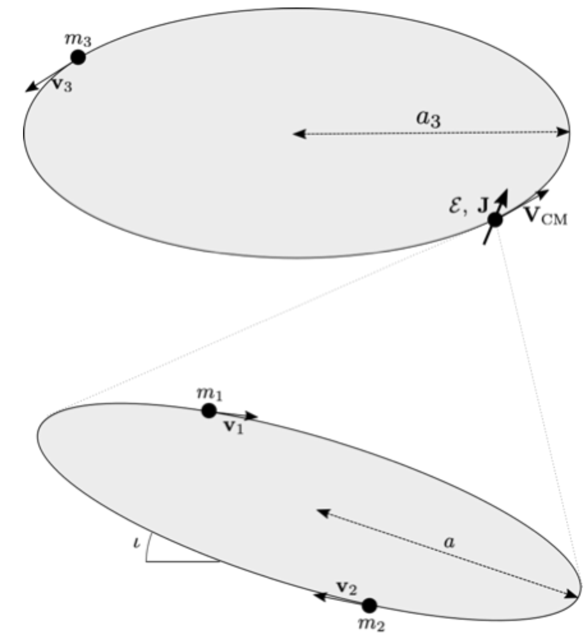

- Lack of physical intuition
- Numerical evolution over long timescales difficult
- Issues in the radiative sector

# EFFECTIVE TWO-BODY

AK, F. Serra, E. Trincherini 2021




3-body motion = 2-body with spin!



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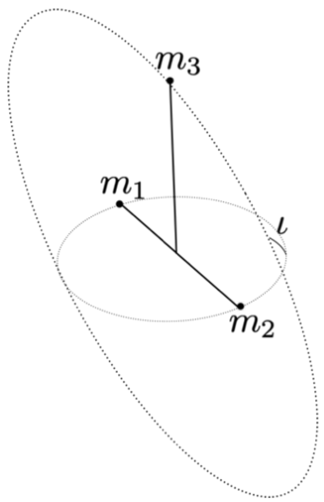


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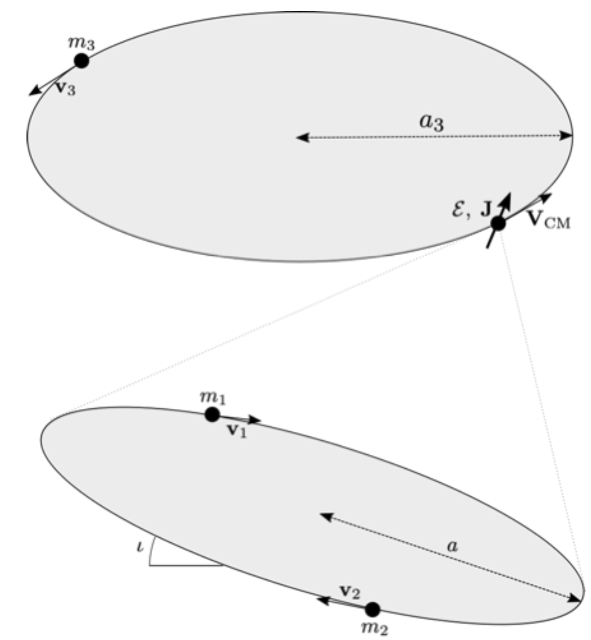


$$\mathscr{L}_{\rm full} = \sum_{A=1}^{3} - m_A \sqrt{-g_{\mu\nu} v_A^\mu v_A^\nu} \quad \Leftrightarrow \quad \mathscr{L}_{\rm EFT} = -\mathscr{E}\sqrt{-g_{\mu\nu} V^\mu_{\rm CM} V^\nu_{\rm CM}} + \frac{1}{2} J_{\mu\nu}\Omega^{\mu\nu} - m_3 \sqrt{-g_{\mu\nu} v_3^\mu v_3^\nu}$$

Proper time

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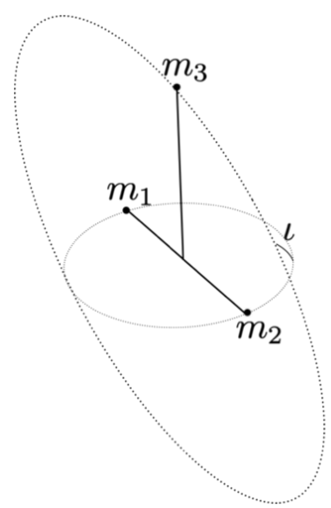


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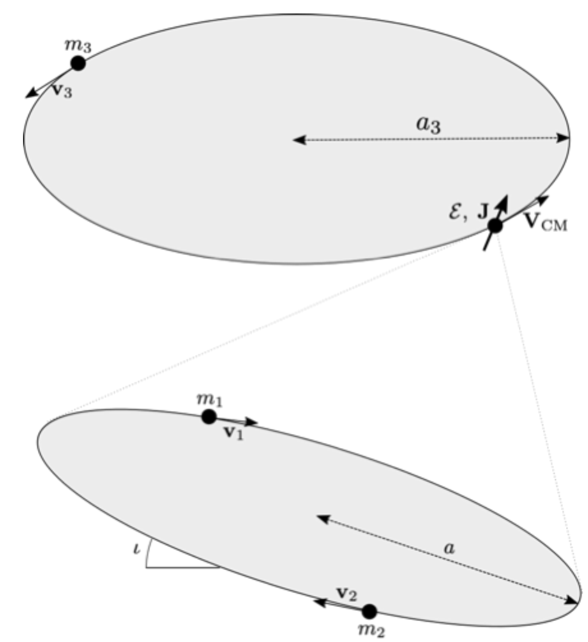


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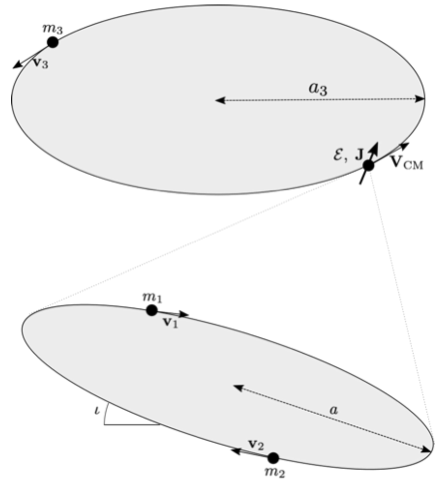
Proper time

The equivalence principle fixes nearly everything!

$$\mathscr{E} = m - \frac{G_N m\mu}{2a}\,, \qquad J_{ij} = \epsilon_{ijk}J^k\,, \quad \Omega_{ij} = \epsilon_{ijk}\Omega^k\,, \qquad \mathbf{J} = \sqrt{G_N m a(1-e^2)}\;\hat{\mathbf{j}}\,, \qquad \mathbf{\Omega} = \hat{\mathbf{e}} \times \dot{\hat{\mathbf{e}}}$$

$\hat{\mathbf{e}} \equiv$ Unit Runge-Lenz vector




# A SYSTEMATIC EXPANSION

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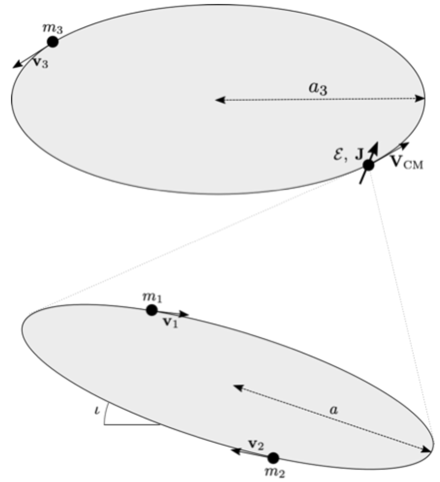
As in any EFT, the Lagrangian is organised with power-counting rules:

$$v^2 \equiv \frac{Gm}{a}$$ and $$\varepsilon \equiv \frac{a}{a_3}$$

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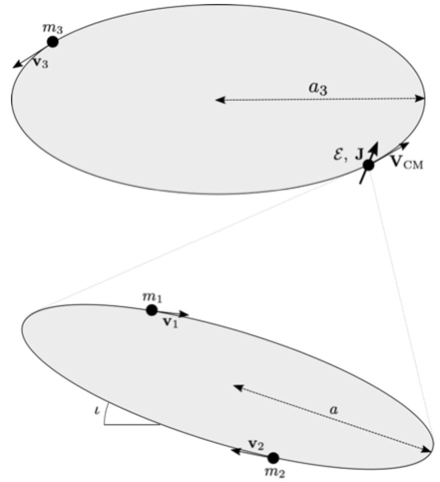
To get the EOM for the point-particles, one should 'integrate out' the gravitational field

$$\mathcal{H} = -\frac{Gm_1m_2}{2a} - 3m\frac{G^2m_1m_2}{a^2\sqrt{1-e^2}}$$

Hamiltonian of inner orbit $\quad \varepsilon^{-1}v^0 + \varepsilon^{-1}v^2$



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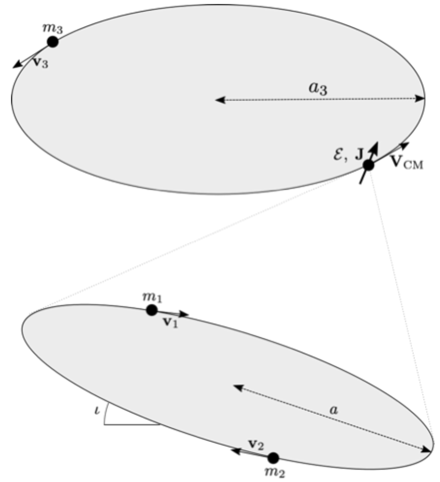


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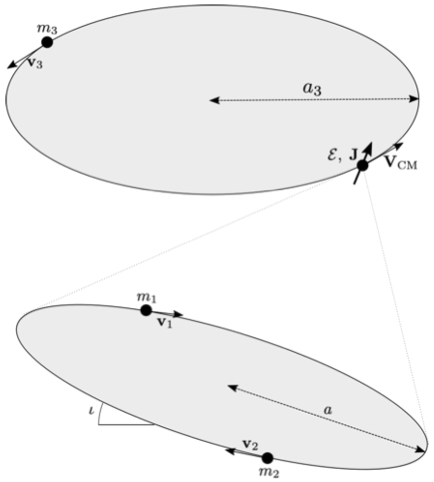
Hamiltonian of outer orbit $\varepsilon^{0}v^0 + \varepsilon^{1}v^2$

$$+\frac{4m+3m_3}{2m}\frac{G}{a_3^3(1-e_3^2)^{3/2}}\mathbf{J}\cdot\mathbf{J}_3$$

Spin-orbit coupling $\varepsilon^{3/2}v^2$

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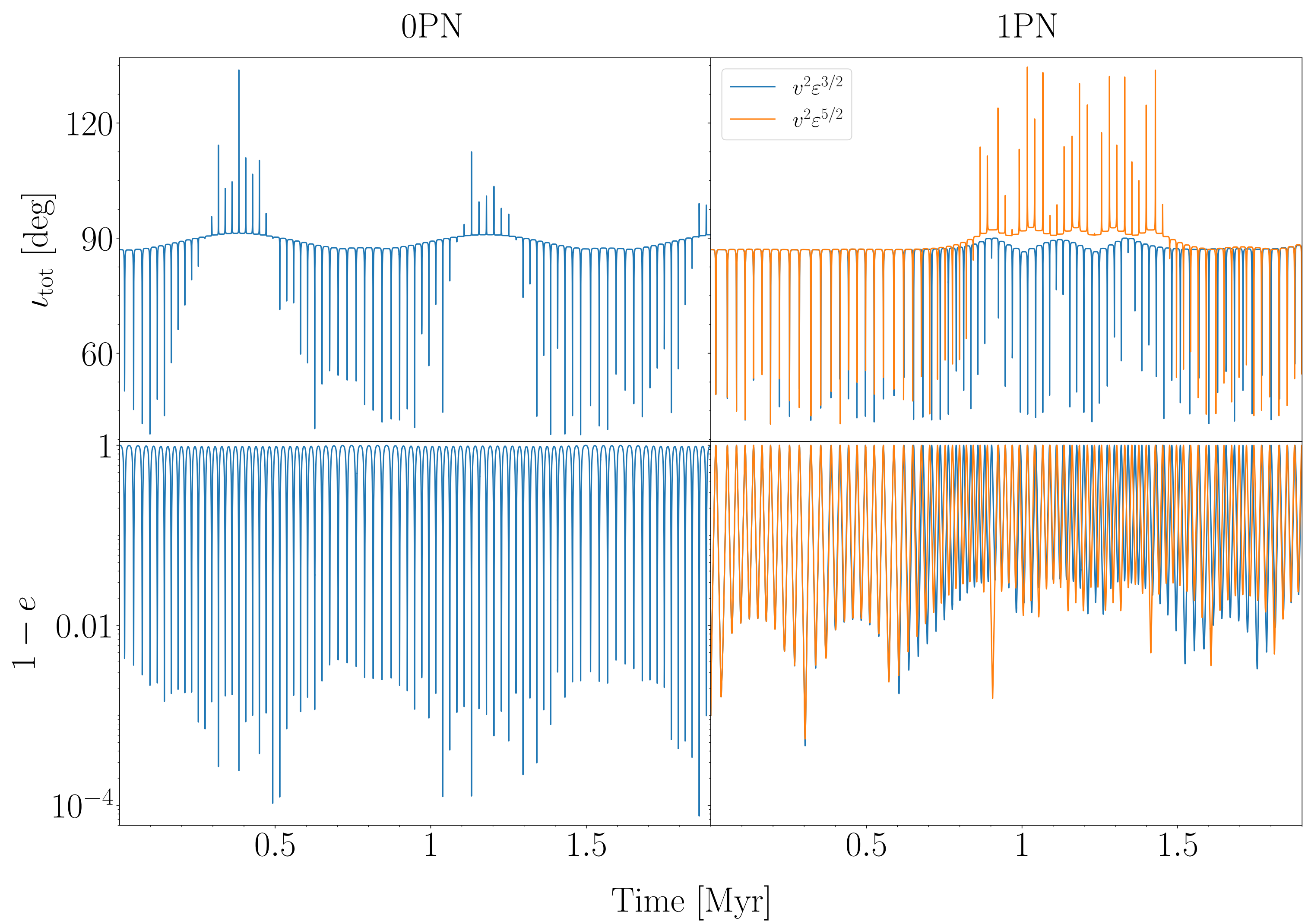
| $-\dfrac{Gmm_3}{2a_3} - 3M\dfrac{G^2mm_3}{a_3^2\sqrt{1-e_3^2}}$ | Hamiltonian of outer orbit | $\varepsilon^{0}v^0 + \varepsilon^{1}v^2$ |
| $+\dfrac{4m+3m_3}{2m}\dfrac{G}{a_3^3(1-e_3^2)^{3/2}}\mathbf{J}\cdot\mathbf{J}_3$ | Spin-orbit coupling | $\varepsilon^{3/2}v^2$ |
| $+\mathcal{H}_{\rm quad}$ | Quadrupolar coupling | $\varepsilon^{2}v^0 + \varepsilon^{2}v^2$ |

# Application #1

## Long-term evolution of relativistic 3-body systems

$$\mathcal{H} = \mathcal{H}_{\text{inner}} + \mathcal{H}_{\text{outer}} + \mathcal{H}_{\varepsilon^{3/2}\upsilon^{2}} + \mathcal{H}_{\varepsilon^{2}\upsilon^{0}} + \mathcal{H}_{\varepsilon^{2}\upsilon^{2}} + \ldots$$




AK, F. Serra, E. Trincherini (In prep.)

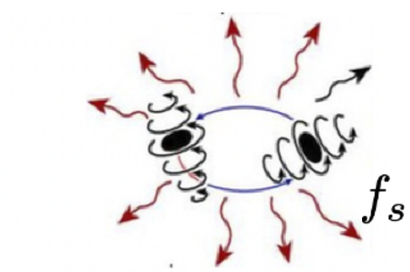
# APPLICATION #2

## RELATIVISTIC DOPPLER EFFECT IN WAVEFORMS

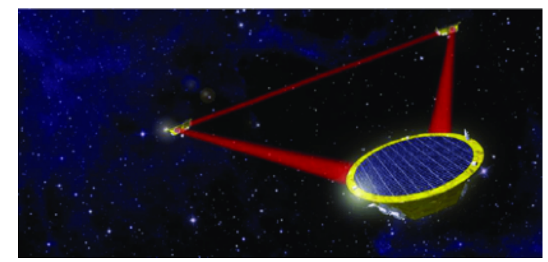
- Longitudinal Doppler effect: Randall Xianyu '18 Inayoshi et al. '17 Strokov et al. '17…

$M$

$$V_{||} = \mathbf{V} \cdot \mathbf{n}$$



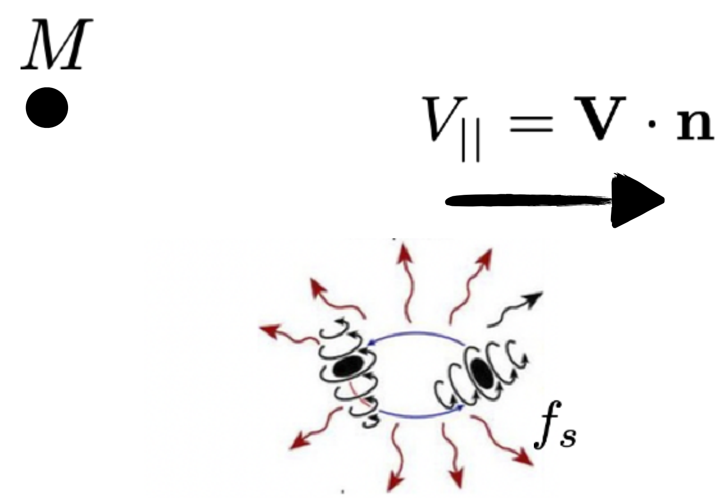

$$f_r = \frac{f_s}{1 + V_{||}}$$



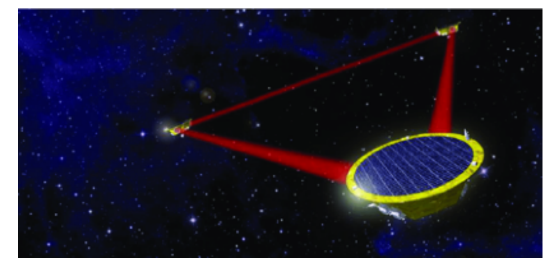
# Application #2

## Relativistic Doppler effect in waveforms

- Longitudinal Doppler effect: Randall Xianyu '18 Inayoshi et al. '17 Strokov et al. '17…




$$f_r = \frac{f_s}{1 + V_{||}}$$



- Transverse Doppler effect: break degeneracies

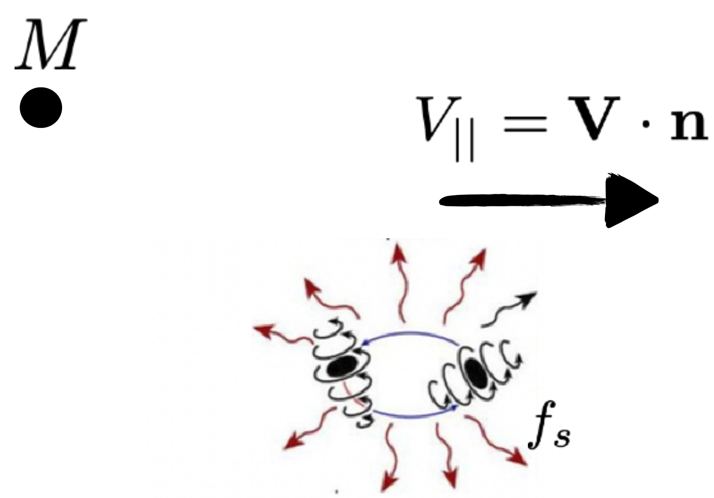
$$f_r = \frac{f_s \sqrt{1 - V^2}}{1 + V_{||}}$$

AK, K. Leyde (In prep.)

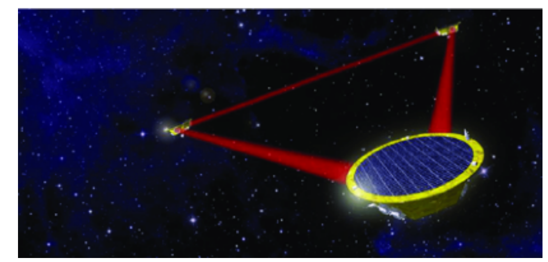
# APPLICATION #2

## RELATIVISTIC DOPPLER EFFECT IN WAVEFORMS

- Longitudinal Doppler effect: Randall Xianyu '18 Inayoshi et al. '17 Strokov et al. '17…




$$f_r = \frac{f_s}{1 + V_{||}}$$

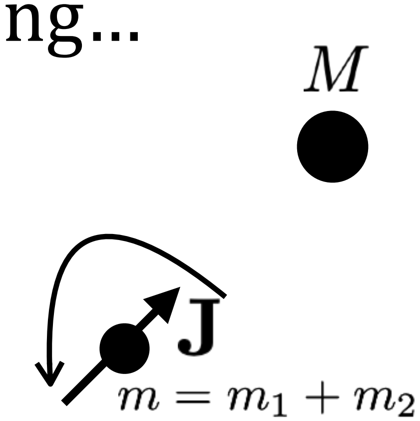


- Transverse Doppler effect: break degeneracies

$$f_r = \frac{f_s\sqrt{1 - V^2}}{1 + V_{||}}$$

AK, K. Leyde (In prep.)

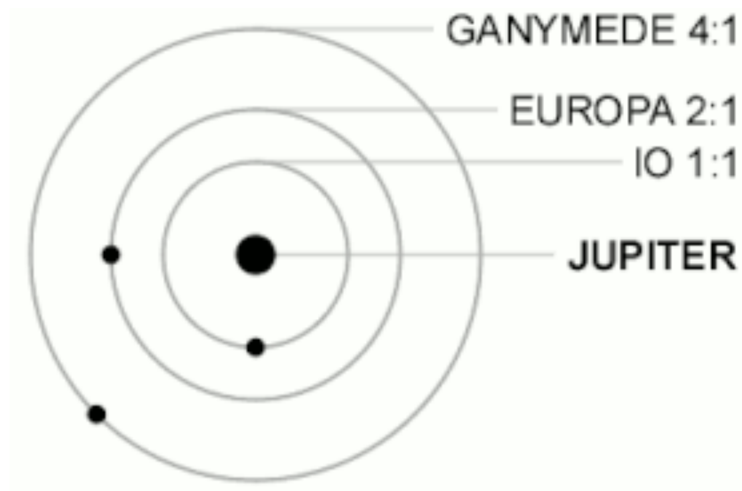
- Higher order effects in waveforms like spin-orbit coupling…



# APPLICATION #3

## NEW RESONANCES

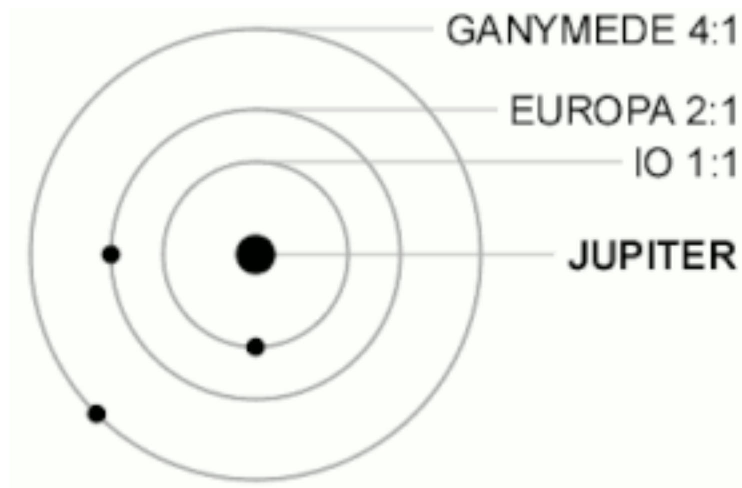
- Resonances are a fascinating phenomenon of the 3-body problem



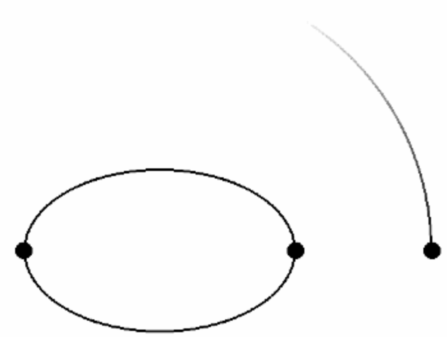
# Application #3

## New resonances

- Resonances are a fascinating phenomenon of the 3-body problem




- When relativistic effects are included, there are other kinds of resonances



Perihelion angle

Outer orbit frequency

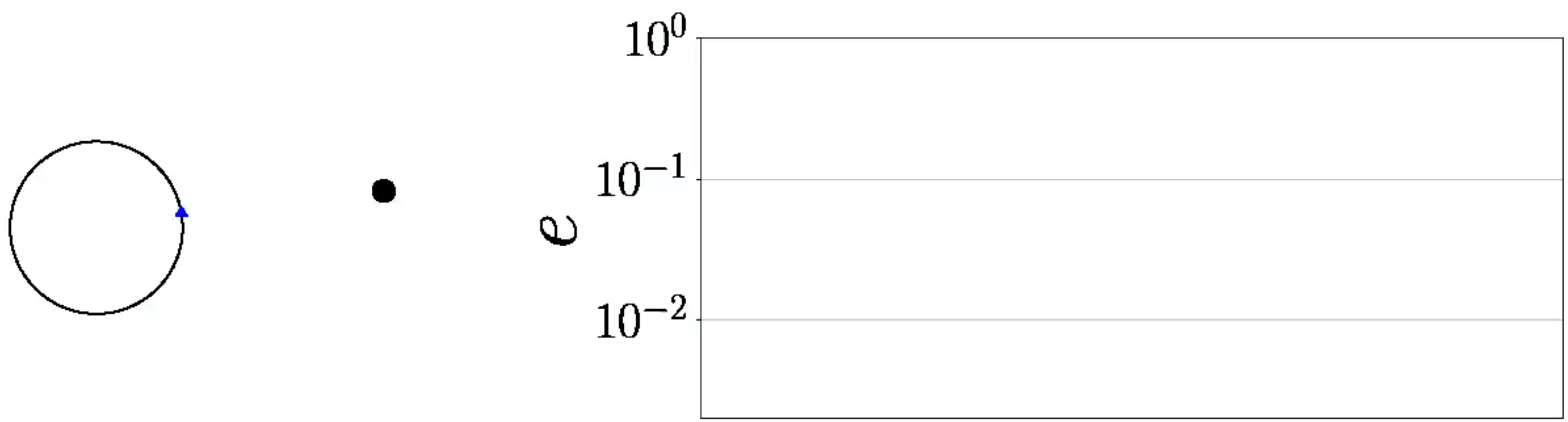
$$p\,\dot{\omega} + q\sqrt{\frac{GM}{a_3^3}} = 0$$

$$p, q \in \mathbb{Z}$$

# APPLICATION #3

## NEW RESONANCES

$$a(t) = a_0 \left(1 - \frac{t}{t_{\mathrm{RR}}}\right)^{1/4}$$




AK 2021

# CONCLUSIONS

- Very rich phenomenology in the Newtonian 3-body problem, even more in the relativistic one…

- EFT formulation suited to precision computations

- Future work: more precise waveforms for 3-body problem