High T QCD Plasma and QCD Phase Diagram

M. Stephanov

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"Simple" and elegant theory based on the gauge principle:

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  - Asymptotic freedom: certain short distance (hard) procesess do reveal quarks and gluons. $\Lambda_{\mathrm{QCD}} = \mathcal{O}(200\,\mathrm{MeV})$, i.e., $1\ \mathrm{fm}^{-1}$.
  - Chiral symmetry breaking: axial symmetry of the Lagrangian emerging in $m_f \to 0$ limit is not realized linearly in the spectrum. Instead, pion is a would-be Goldstone boson.

QCD in a many-body environment

- Examples: neutron stars, sub-ms-old Universe, heavy-ion collisions. Characteristic energy per d.o.f. of order Λ_{QCD}.

QCD in a many-body environment

- Examples: neutron stars, sub-ms-old Universe, heavy-ion collisions. Characteristic energy per d.o.f. of order $\Lambda_{\rm QCD}$.
- Statistical approach. First approximation is always equilibrium. I.e., the most “likely” ensemble of quantum states of the system at given total energy.
- Characterized by temperature: energy per d.o.f. (energy needed to increase entropy by 1.)

Thermodynamics

- The partition function:

$$Z(T) = \sum_{\text{all states}} e^{-E_{\text{state}}/T} = \operatorname{Tr} e^{-H/T}.$$

can be calculated by a path integral in Euclidean space with compactified ($\beta = 1/T$) imaginary time direction.

Pressure: $p(T) = (T/V) \log Z$.

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- At low T ($\ll \Lambda_{\text{QCD}}$) hot QCD is a gas of pions, with a little bit of baryons. Interacting via resonances – *hadron resonance gas*.

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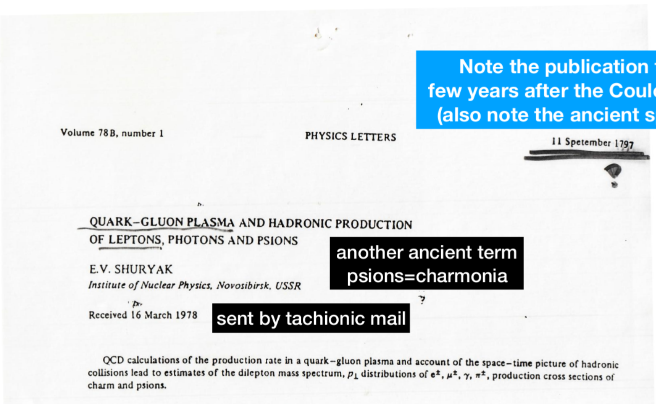
E. Shuryak @ 40 years of QGP in Wuhan, 2018


Volume 78B, number 1 PHYSICS LETTERS 11 Spetember 1797

QUARK–GLUON PLASMA AND HADRONIC PRODUCTION OF LEPTONS, PHOTONS AND PSIONS

E.V. SHURYAK

Institute of Nuclear Physics, Novosibirsk, USSR

Received 16 March 1978

QCD calculations of the production rate in a quark–gluon plasma and account of the space–time picture of hadronic collisions lead to estimates of the dilepton mass spectrum, $p_\perp$ distributions of $e^\pm$, $\mu^\pm$, γ, $\pi^\pm$, production cross sections of charm and psions.

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- Most commonly: a discontinuity of ε: $\varepsilon(T_c - 0) \neq \varepsilon(T_c + 0)$.
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- In pure YM, there is a (global discrete Z_{N_c}) symmetry of Euclidean QFT which breaks above a certain $T_c = \mathcal{O}(\Lambda_{\rm QCD})$ (Polyakov 1977).

  The order parameter is Polyakov line: $\langle \mathrm{Tr}\, P \exp(i \int A_t dt\rangle = e^{-\beta F_q}$. Vanishing below T_c signifies confinement ($F_q = \infty$).

- For $N_c = 3$ this transition is first-order.

YM + quarks = QCD

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- Quarks break Z_3 symmetry. Polyakov order parameter is always nonzero in QCD.
- Linear string-like confining potential is also not possible due to string (flux-tube) breaking by $q\bar{q}$.
- Asymptotic freedom still suggests that at high T – Quark-Gluon Plasma.
- Is there a phase transition from Hadron Gas to Quark-Gluon Plasma?

The chiral symmetry

- For massless quarks, QCD has global axial (chiral) symmetry $SU_A(N_f)$. It is spontaneously broken in the vacuum.
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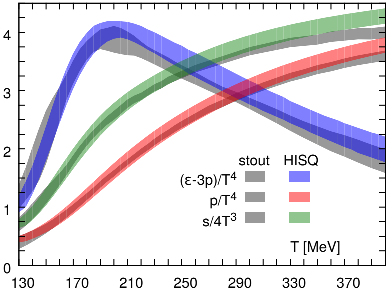
- For massless quarks, QCD has global axial (chiral) symmetry $SU_A(N_f)$. It is spontaneously broken in the vacuum.
- If this symmetry is restored in QGP there *must* be a phase transition. What is the order?
- Generically – first order. But for $N_f = 2$ the second-order transition is possible (Pisarski-Wilczek 1983).

  Because a conformal theory (correlation length $\to \infty$) with one relevant operator (tuned by T) exists for the corresponding universality class $SU_A(2) \times SU_V(2) = O(4)$.
- But the quarks are not massless.

Lattice

- First-principle calculation of QCD partition function at finite T. On a discretized, finite volume space-time this becomes a problem solvable by Monte Carlo methods.

- Lattice calculations reveal that there is a crossover, not a phase transition for physical (finite) quark masses.



- At asymptotically large T pressure, entropy, etc approach the QGP Stefan-Boltzmann limit.

“Perfect” fluid

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- Viscosity in a gas $\eta \sim \ell_{\rm m.f.p.} \sim 1/{\rm coupling}^2$ and $\eta/\hbar s$ is large. Lattice calculations (notoriosly difficult in this case) and heavy-ion collision measurements indicate that $\eta/\hbar s \sim 0.2$ near T_c.

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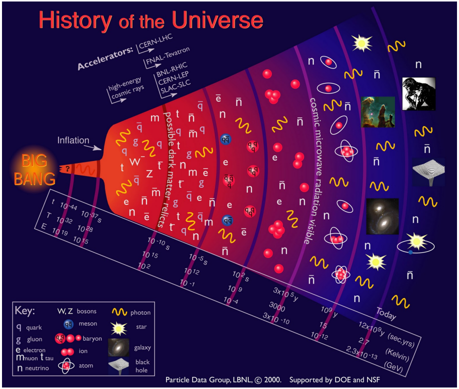
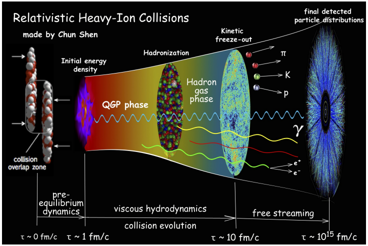
- In a special set of infinitely strongly coupled theories $\eta/\hbar s = 1/4\pi$ is the lower bound.

  Heavy-ion collision experiments indicate that sQGP may be saturating it.

Big Bang vs little bangs



Little Bang



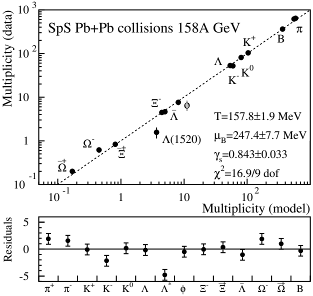
Expansion accompanied by cooling, followed by freezeout.

- Difference: space itself not expanding in HIC.
- Difference: One Event vs many events (cosmic variance vs e.b.e. fluctuations)
- Difference: tunable parameter – $\sqrt{s}$.

Heavy-Ion Collisions. Thermalization.

"Little Bang"

- The final state looks thermal.
- Similar to CMB.



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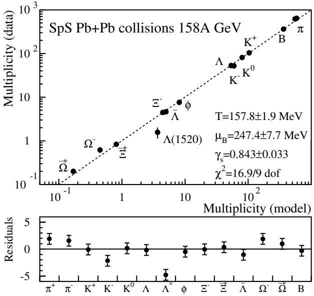

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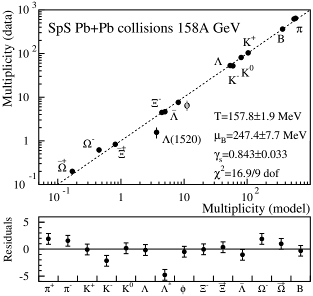

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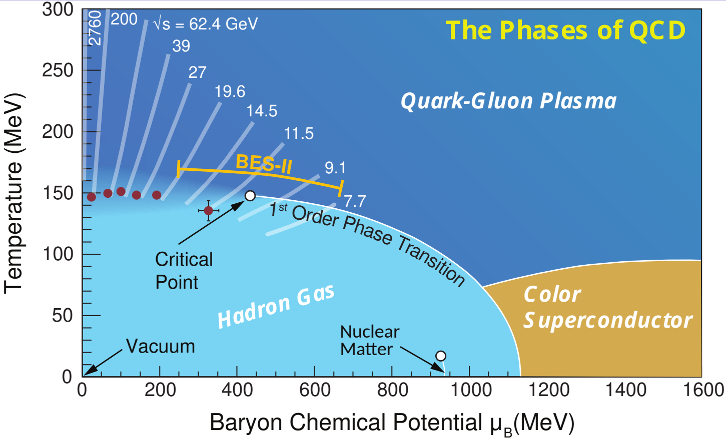
- Flow – looks hydrodynamic. Initial anisotropy fluctuations are propagated to final state hydrodynamically.
- $\sqrt{s}$ controls baryon asymmetry in the final state. Quantified by baryon chemical potential μ_B – energy cost of adding a baryon.

Phase diagram of QCD



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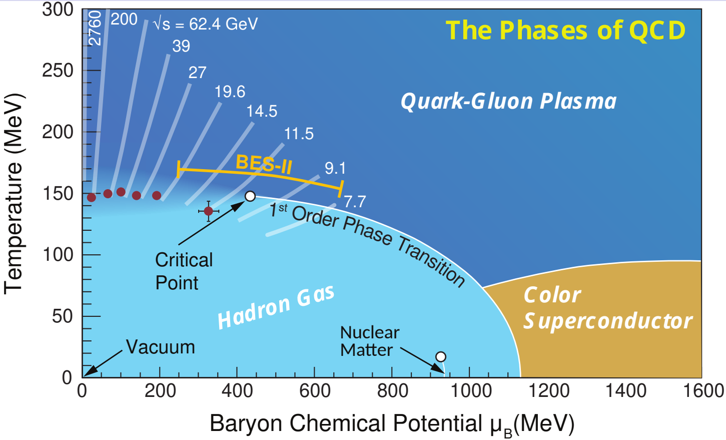
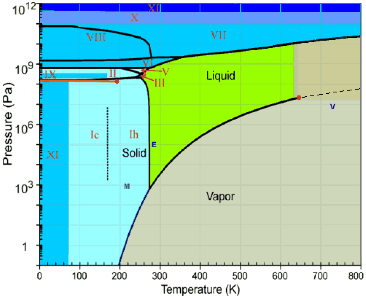


- Heavy-ion collisions explore QCD phase diagram by varying $\sqrt{s}$.
- Does the transition become first-order at some μ_B?
- Lattice calculations at finite μ_B, despite recent progress, are still hindered by sign problem.

Substance[13][14]	Critical temperature	Critical pressure (absolute)
Argon	−122.4 °C (150.8 K)	48.1 atm (4,870 kPa)
Ammonia[15]	132.4 °C (405.5 K)	111.3 atm (11,280 kPa)
Bromine	310.8 °C (584.0 K)	102 atm (10,300 kPa)
Caesium	1,664.85 °C (1,938.00 K)	94 atm (9,500 kPa)
Chlorine	143.8 °C (416.9 K)	76.0 atm (7,700 kPa)
Ethanol	241 °C (514 K)	62.18 atm (6,300 kPa)
Fluorine	−128.85 °C (144.30 K)	51.5 atm (5,220 kPa)
Helium	−267.96 °C (5.19 K)	2.24 atm (227 kPa)
Hydrogen	−239.95 °C (33.20 K)	12.8 atm (1,300 kPa)
Krypton	−63.8 °C (209.3 K)	54.3 atm (5,500 kPa)
CH_4 (methane)	−82.3 °C (190.8 K)	45.79 atm (4,640 kPa)
Neon	−228.75 °C (44.40 K)	27.2 atm (2,760 kPa)
Nitrogen	−146.9 °C (126.2 K)	33.5 atm (3,390 kPa)
Oxygen	−118.6 °C (154.6 K)	49.8 atm (5,050 kPa)
CO_2	31.04 °C (304.19 K)	72.8 atm (7,380 kPa)
N_2O	36.4 °C (309.5 K)	71.5 atm (7,240 kPa)
H_2SO_4	654 °C (927 K)	45.4 atm (4,600 kPa)
Xenon	16.6 °C (289.8 K)	57.6 atm (5,840 kPa)
Lithium	2,950 °C (3,220 K)	652 atm (66,100 kPa)
Mercury	1,476.9 °C (1,750.1 K)	1,720 atm (174,000 kPa)
Sulfur	1,040.85 °C (1,314.00 K)	207 atm (21,000 kPa)
Iron	8,227 °C (8,500 K)	
Gold	6,977 °C (7,250 K)	5,000 atm (510,000 kPa)
Water[2][16]	373.946 °C (647.096 K)	217.7 atm (22.06 MPa)

Critical point
– end of phase coexistence –
is a ubiquitous phenomenon

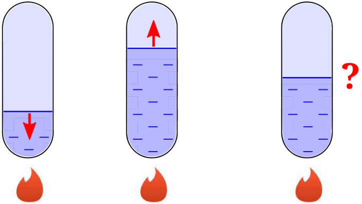
Water:



Is there one in QCD?

History

Cagniard de la Tour (1822): discovered continuos transition from liquid to vapour by heating alcohol, water, etc. in a gun barrel, glass tubes.



Name

Faraday (1844) – liquefying gases:

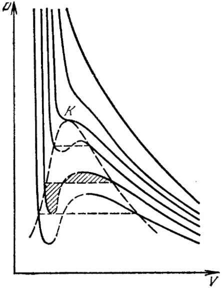
"Cagniard de la Tour made an experiment some years ago which gave me occasion to want a new word."

Mendeleev (1860) – measured vanishing of liquid-vapour surface tension: "Absolute boiling temperature".

Andrews (1869) – systematic studies of many substances established continuity of vapour-liquid phases. Coined the name "critical point".

Theory

van der Waals (1879) –
in “On the continuity of the gas and liquid state”
(PhD thesis) wrote e.o.s. with a critical point.



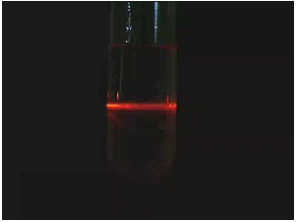
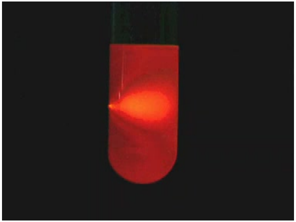
Smoluchowski, Einstein (1908,1910) – explained critical opalescence.

Landau – classical theory of critical phenomena

Fisher, Kadanoff, Wilson – scaling, full fluctuation theory based on RG.

Critical opalescence

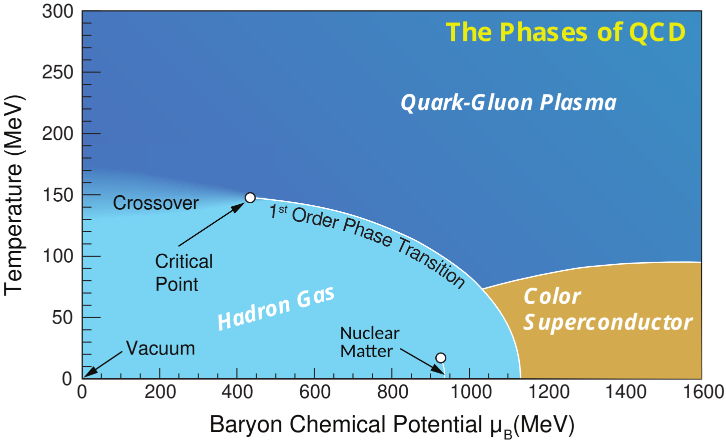
shining laser light through liquid

Critical point between the QGP and hadron gas phases?

QCD is a relativistic theory of a fundamental force.

CP is a singularity of EOS, anchors the 1st order transition.



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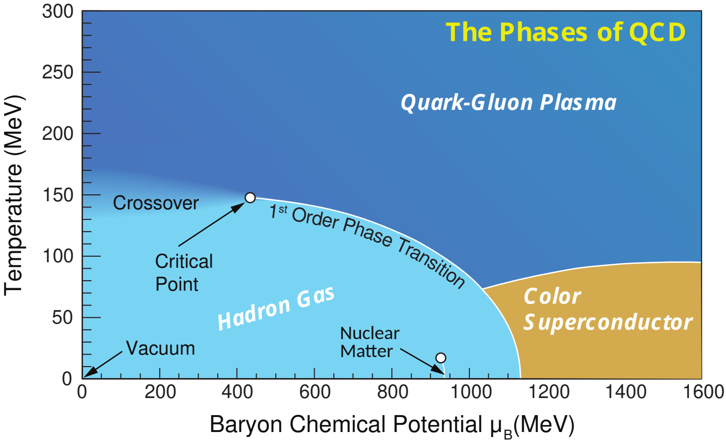

Lattice QCD at $\mu_B \lesssim 2T$ – a crossover.

C.P. is ubiquitous in models (NJL, RM, Holog., Strong coupl. LQCD, . . .)

Assumption for the next part of this talk

H.I.C. are sufficiently close to equilibrium that we can study thermodynamics at freezeout T and μ_B — as a first approximation.

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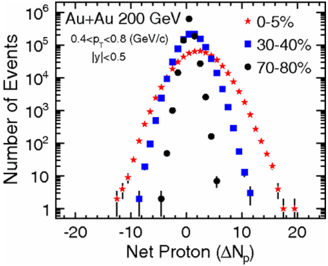
- NB: Event-by-event fluctuations:

  Heavy-ion collisions create systems which are large enough (for thermodynamics), but not too large ($N \sim 10^2 - 10^4$ particles)

  EBE fluctuations are small ($1/\sqrt{N}$), but measurable.



What are the signatures of the critical point?

EBE fluctuations vs $\sqrt{s}$ [PRL81(1998)4816]

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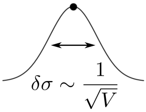

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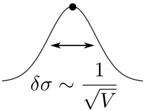
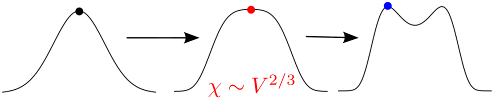


- At the critical point $S(\sigma)$ "flattens". And $\chi \equiv \langle \delta\sigma^2 \rangle V \to \infty$.



CLT?

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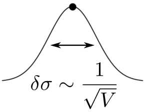

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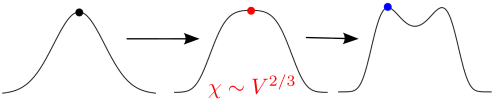
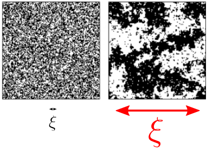




CLT?

$\delta\sigma$ is not an average of ∞ many *uncorrelated* contributions: $\xi \to \infty$

In fact, $\langle \delta\sigma^2 \rangle \sim \xi^2 / V$.

Higher order cumulants

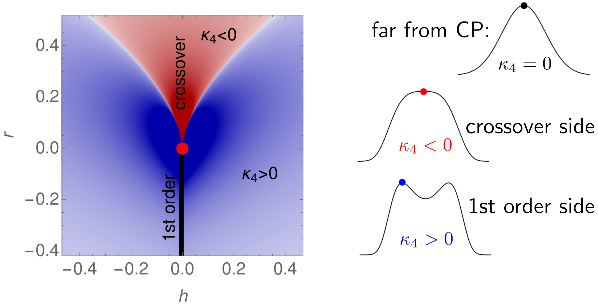
- $n > 2$ cumulants (shape of $P(\sigma)$) depend stronger on ξ.

  E.g., $\langle\sigma^2\rangle \sim \xi^2$ while $\kappa_4 = \langle\sigma^4\rangle_c \sim \xi^7$ *[PRL102(2009)032301]*

- For $n > 2$, sign depends on which side of the CP we are.

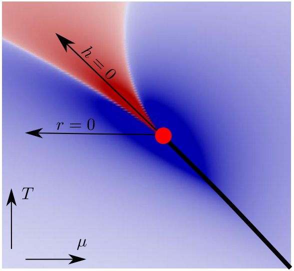
  This dependence is also universal. *[PRL107(2011)052301]*

- Using Ising model variables:



Mapping Ising to QCD and observables near CP

κ_4 vs μ_B and T:



- In QCD $(r, h) \to (\mu - \mu_{\text{CP}}, T - T_{\text{CP}})$

Rehr-Mermin, 1973

Parotto *et al*, 1805.05249

Pradeep-MS, 1905.13247

Mrozcek *et al*, 2008.04022

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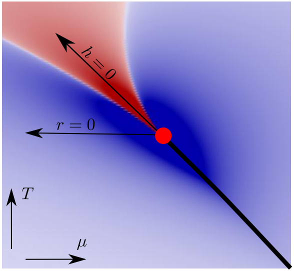

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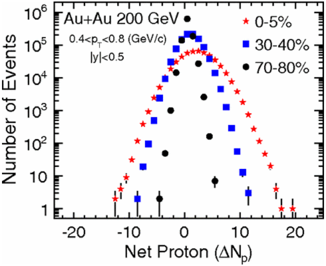


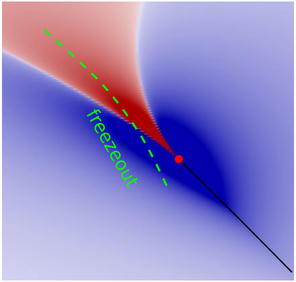
- Experiments do not measure σ. Fluctuations of σ are "imprinted" on hadron multiplicities.

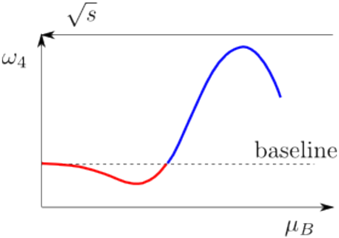
$$\kappa_n(N) = N + \mathcal{O}(\kappa_n(\sigma))$$ MS, 1104.1627

Pradeep et al 2109.13188

Beam Energy Scan I: intriguing hints

Equilibrium κ_4 vs μ_B and T:





Beam Energy Scan I: intriguing hints

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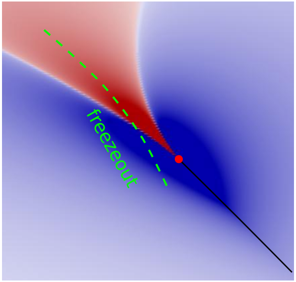

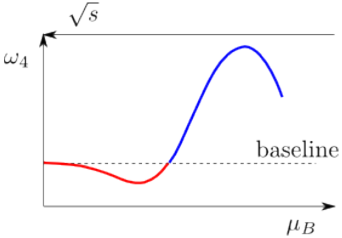

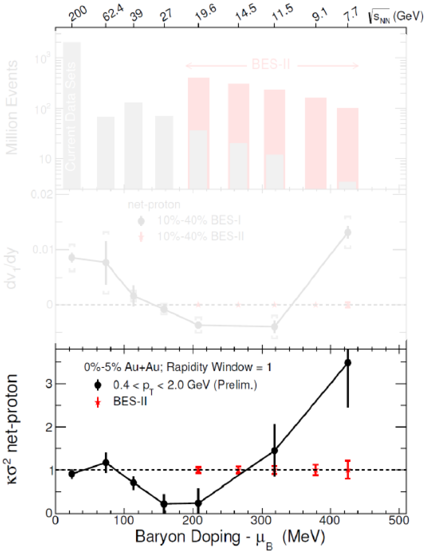

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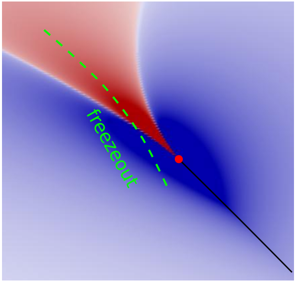

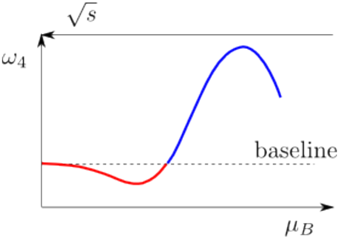

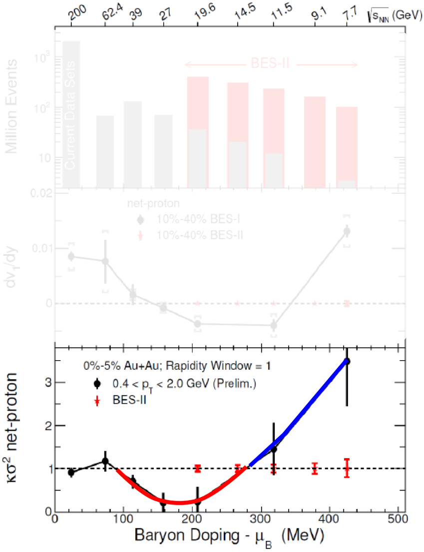

"intriguing hint" (2015 LRPNS)

Beam Energy Scan I: intriguing hints

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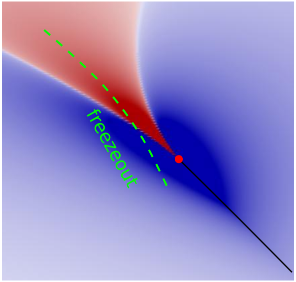

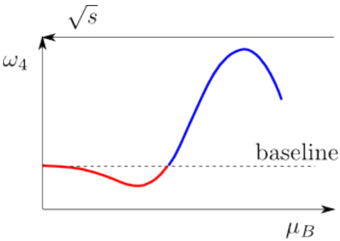

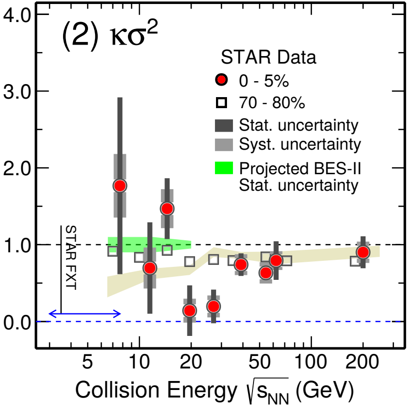

STAR 2001.02852

"non-monotonic with 3.1σ significance"

Beam Energy Scan I: intriguing hints

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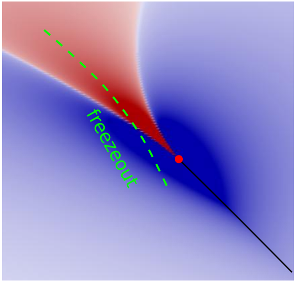

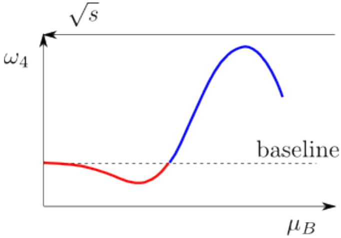

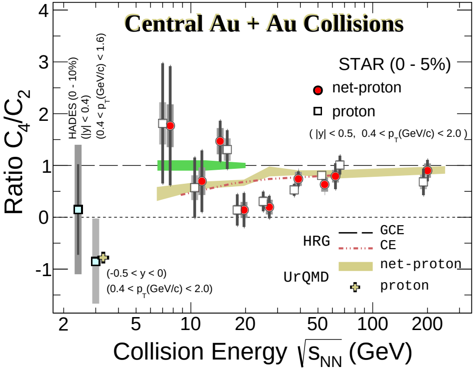

STAR 2112.00240

Theory/experiment gap: predictions assume equilibrium, but
in heavy-ion collisions

near the critical point non-equilibrium physics is essential.

Because of the critical slowing down, certain slow degrees of freedom are further away from equilibrium. These degrees of freedom are directly related to fluctuations.

Challenge: develop hydrodynamics *with fluctuations* capable of describing *non-equilibrium* effects on critical-point signatures.

Randomness is hydrodynamics

- Hydrodynamics is a coarse-grained theory. Relies on scale separation: $\ell_{\rm wave} \gg \ell_{\rm mic}$, $\tau_{\rm evolution} \gg \tau_{\rm equilibration}$.

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- Operators T^{i0}, J^0 *coarse-grained* over "hydrodynamic cells" b, $\ell_{\rm wave} \gg b \gg \ell_{\rm mic}$, are stochastic variables and obey

$$\partial_t \breve{\psi} = -\nabla \cdot \Big(\text{Flux}[\breve{\psi}] + \text{Noise} \Big) \qquad \textit{(Landau-Lifshitz)}$$

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- Non-linearities + locality $\Rightarrow$ UV divergences, "long-time tails". In numerical simulations – cutoff dependence.

Randomness in hydrodynamics

Stochastic description

Random hydro variables: $\breve{\psi}$

$$\partial_t \breve{\psi} = -\nabla \cdot \Big(\text{Flux}[\breve{\psi}] + \text{Noise} \Big)$$

$+$ fewer variables and eqs.

$-$ stochastic

$-$ cutoff dependence

Landau-Lifshits, Kapusta et al, Gale et al, Nahrgang et al, . . .

Deterministic description

$\psi \equiv \langle \breve{\psi} \rangle$, $G \equiv \langle \breve{\psi} \breve{\psi} \rangle$, etc.

$$\partial_t \psi = -\nabla \cdot \text{Flux}[\psi; G];$$

$$\partial_t G = -2\Gamma(G - \bar{G}[\psi]);$$

$-$ more variables and eqs.

$+$ deterministic

$+$ no cutoff dependence after renormalization

Andreev, Akamatsu et al, Yin et al, An et al, Martinez et al, . . .

Summary

- Is there a critical point between QGP and hadron gas phases?

  Heavy-Ion collision experiments may answer.

  The quest for the QCD critical point challenges us to creatively apply existing concepts and develop new ideas.

- Large (non-gaussian) fluctuations – universal signature of a critical point.

- In H.I.C., the magnitude of the signatures is controlled by non-equilibrium effects. The interplay of critical phenomena and non-equilibrium dynamics opens interesting questions.

More

Hydro+

Yin, MS, 1712.10305

- Hydro+ extends Hydro with new *non-hydrodynamic* d.o.f..
- At the CP, the *slowest* (i.e., most out of equilibrium) new d.o.f. is the 2-pt function $\langle \delta m \delta m \rangle$ of the slowest hydro variable $m \equiv s/n$:

$$\phi_{\boldsymbol{Q}}(\boldsymbol{x}) = \int_{\Delta \boldsymbol{x}} \langle \delta m\,(\boldsymbol{x}_1)\ \delta m\,(\boldsymbol{x}_2) \rangle\ e^{i\boldsymbol{Q}\cdot\Delta\boldsymbol{x}}$$

where $\boldsymbol{x} = (\boldsymbol{x}_1 + \boldsymbol{x}_2)/2$ and $\Delta \boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$.

- In equilibrium fluctuations are determined by thermodynamics:

$$\bar{\phi}_{\boldsymbol{Q}} = \frac{c_p}{n^2} f(\boldsymbol{Q}) \approx \frac{c_p}{n^2} \frac{1}{1 + \boldsymbol{Q}^2 \xi^2}\,.$$

Relaxation of fluctuations towards equilibrium

- As usual, equilibration maximizes entropy $S = \sum_i p_i \log(1/p_i)$:

$$s_{(+)}(\epsilon, n, \phi_{\boldsymbol{Q}}) = s(\epsilon, n) + \frac{1}{2}\int_{\boldsymbol{Q}} \left(\log \frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}} - \frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}} + 1 \right)$$

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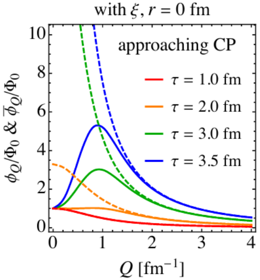
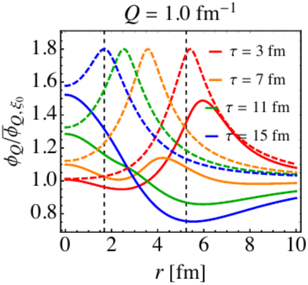
- The equation for $\phi_{\boldsymbol{Q}}$ is a relaxation equation with rate

$$\Gamma(\boldsymbol{Q}) \approx 2DQ^2 \quad \text{for} \quad Q \ll \xi^{-1}, \quad D \sim 1/\xi.$$

- Impact on fluctuation observables: critical slowing down, “memory” effects *(Berdnikov-Rajagopal, Mukherjee-Venugopalan-Yin, . . .)*
- Impact of fluctuations on hydrodynamics:
  - “Renormalization” of bulk viscosity $\zeta \sim 1/\Gamma_\xi \sim \xi^3$.
  - (Non-analytic) frequency dependence of $\zeta(\omega)$ for $\omega \ll \Gamma_\xi$.
    “Long-time tails”

Implementation of Hydro+ and lessons





Rajagopal et al, 1908.08539
Du et al, 2004.02719

- Conservation laws
- Memory and lag
- Advection
- Feedback is small

General covariant formalism

An, Basar, Yee, MS, 1902.09517,1912.13456

- To embed Hydro+ into a unified theory for critical as well as non-critical fluctuations we need a general *deterministic* (*hydro-kinetic*) formalism.
- Expand stochastic hydro eqs. in $\{\delta m, \delta p, \delta u^\mu\} \sim \phi_A$ and then average, using equal-time correlator as a new variable

$$G_{AB}(x_1, x_2) \stackrel{?}{=} \langle\, \phi_A(x_1)\, \phi_B(x_2)\, \rangle.$$

  - What is “equal-time” in *relativistic* hydro?
  - $\langle \phi(x)\phi(x) \rangle$ is singular (cutoff dependent). Renormalization?

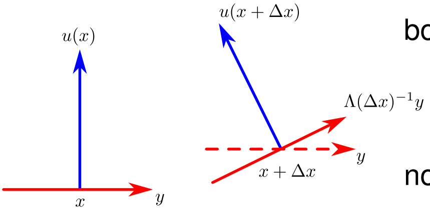
Local equal time and *confluent* derivative

- We need equal-time correlator $G = \langle \phi(t, \boldsymbol{x}_1) \phi(t, \boldsymbol{x}_2) \rangle$.

  But what does "equal time" mean? In what frame?

  The most natural choice is local $u(x)$ (at $x = (x_1 + x_2)/2$).

- x-derivative with $y \equiv x_1 - x_2$ "fixed" w.r.t. local rest frame:



boost $\Lambda(\Delta x) u(x + \Delta x) = u(x)$:

$$\Delta x \cdot \bar{\nabla} G(x; y) \equiv$$
$$G(x + \Delta x; \Lambda(\Delta x)^{-1} y) - G(x, y)\,.$$

not $G(x + \Delta x; y) - G(x; y)\,.$

- We define *confluent* equal time correlator $\bar{G}_{AB}(x; y)$ and its Wigner transform $W_{AB}(x; q)$

more

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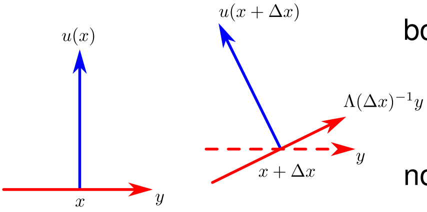

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Renormalization

- Expansion of $\langle T^{\mu\nu} \rangle$ in fluctuations ϕ contains more

$$\langle \phi(x)\phi(x) \rangle = G(x;0) = \int \frac{d^3q}{(2\pi)^3} W(x;q).$$

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- Such short-distance singularities can be absorbed into redefinion of EOS (i.e., pressure) and transport coefficients:

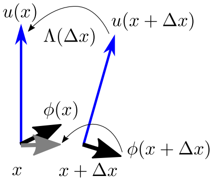
$$\begin{aligned}\langle T^{\mu\nu}(x)\rangle &= \epsilon u^\mu u^\nu + p(\epsilon,n)\Delta^{\mu\nu} + \Pi^{\mu\nu} + \left\{G(x,0)\right\} \\ &= \epsilon_R u^\mu_R u^\nu + p_R(\epsilon_R,n_R)\Delta^{\mu\nu}_R + \Pi^{\mu\nu}_R + \left\{\tilde G(x;0)\right\}\,.\end{aligned}$$

Constraints of 2nd law, conformality satisfied *nontrivially*. more

Confluent derivative, connection and correlator

Take out dependence of *components* of ϕ due to change of $u(x)$:

$$\Delta x \cdot \bar{\nabla}\phi = \Lambda(\Delta x)\phi(x+\Delta x) - \phi(x)$$



Confluent two-point correlator:

$$\bar{G}(x,y) = \Lambda(x_1 - x)\,\langle\phi(x_1)\,\phi(x_2)\rangle\,\Lambda(x_2 - x)^T$$

(boost to $u(x)$ – rest frame at midpoint)

$$\bar{\nabla}_\mu \bar{G}_{AB} = \partial_\mu \bar{G}_{AB} - \bar{\omega}^{C}_{\mu A}\bar{G}_{CB} - \bar{\omega}^{C}_{\mu B}\bar{G}_{AC} - \mathring{\omega}^{b}_{\mu a}\, y^a \frac{\partial}{\partial y^b}\bar{G}_{AB}\,.$$

Connection $\bar{\omega}$ corresponds to the boost Λ.

back

Connection $\mathring{\omega}$ makes sure derivative is independent of the choice of basis triad $\boldsymbol{e}_a(x)$ needed to express $\boldsymbol{y} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2$ in local rest frame.

We then define the Wigner transform $W_{AB}(x;q)$ of $\bar{G}_{AB}(x;y)$.

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Expansion of $\langle T^{\mu\nu}\rangle$ contains $\langle\phi(x)\phi(x)\rangle = G(x;0) = \int \frac{d^3q}{(2\pi)^3}\, W(x;q)$.

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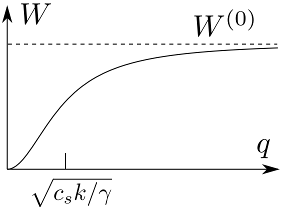

$$W(x,q) \sim \underbrace{W^{(0)}}_{Tw} + \underbrace{W^{(1)}}_{\partial u/q^2} + \widetilde{W}$$

(∼"OPE" or gradient expansion)

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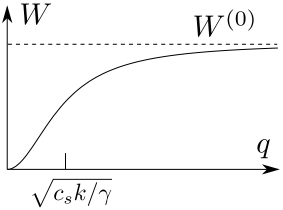

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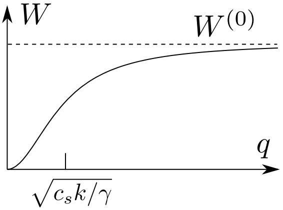

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$$= \epsilon_R u_R^\mu u^\nu + p_R(\epsilon_R,n_R)\Delta_R^{\mu\nu} + \Pi_R^{\mu\nu} + \Big\{\tilde{G}(x,0)\Big\}\,.$$

Renormalized e.o.s. and transport coefficients

- Fluctuation corrections to kinetic coefficients are positive.

back

- Corrections to pressure and bulk viscosity vanish for conformal e.o.s.

$$p_R(\epsilon_R, n_R) = p(\epsilon_R, n_R) + \frac{T\Lambda^3}{6\pi^2}\left((1 - c_s^2 - 2\dot{T} + \dot{c}_s) + \frac{1}{2}(1 - \dot{c}_p)\right),$$

$$\eta_R = \eta + \frac{T\Lambda}{30\pi^2}\left(\frac{1}{\gamma_L} + \frac{7}{2\gamma_\eta}\right),$$

$$\zeta_R = \zeta + \frac{T\Lambda}{18\pi^2}\left(\frac{1}{\gamma_L}(1 - 3\dot{T} + 3\dot{c}_s)^2 + \frac{2}{\gamma_\eta}\left(1 - \frac{3}{2}(\dot{T} + c_s^2)\right)^2 + \frac{9}{4\gamma_\lambda}(1 - \dot{c}_p)^2\right),$$

$$\lambda_R = \lambda + \frac{T^2 n^2 \Lambda}{3\pi^2 w^2}\left(\frac{c_p T}{(\gamma_\eta + \gamma_\lambda) w} + \frac{c_s^2}{2\gamma_L}\right).$$

$$\gamma_\eta \equiv \frac{\eta}{w}, \quad \gamma_\zeta \equiv \frac{\zeta}{w}, \quad \gamma_\lambda \equiv \frac{\kappa}{c_p} = D, \quad \dot{X} \equiv \left(\frac{\partial \log X}{\partial \log s}\right)_m .$$

non-Gaussian fluctuations are sensitive signatures of the critical point

Nonlinearity and multiplicative noise

An et al 2009.10742, PRL

- Stochastic approach (with multiplicative noise):

$$\partial_t \breve{\psi} = -\nabla \cdot \Big(\text{Flux}[\breve{\psi}] + \text{Noise} \Big), \qquad \langle \text{Noise}\,\text{Noise} \rangle \sim 2Q[\breve{\psi}] .$$

- Deterministic approach (Hydro+). *Infinite* hierarchy of coupled equations for cumulants $G_n^{\mathsf{c}} \equiv \langle \delta\breve{\psi} \ldots \delta\breve{\psi} \rangle^{\mathsf{c}}$:

$$\partial_t \psi = -\nabla \cdot \text{Flux}[\psi, G, G_3^{\mathsf{c}}, G_4^{\mathsf{c}}, \ldots];$$

$$\partial_t G = \mathsf{L}[\psi, G, G_3^{\mathsf{c}}, G_4^{\mathsf{c}}, \ldots];$$

$$\partial_t G_3^{\mathsf{c}} = \mathsf{L}_3[\psi, G, G_3^{\mathsf{c}}, G_4^{\mathsf{c}}, \ldots];$$

$$\vdots$$

Controlled perturbation theory

- *Small* fluctuations are *almost* Gaussian
- Introduce expansion parameter ε, so that $\delta\breve{\psi} \sim \sqrt{\varepsilon}$.

  Then $G_n^{\mathsf{c}} \equiv \varepsilon^{n-1}$ and to leading order in ε:

$$\partial_t \psi = -\nabla \cdot \mathsf{Flux}[\psi] + \mathcal{O}(\varepsilon);$$

$$\partial_t G = -2\Gamma(G - \bar{G}[\psi]) + \mathcal{O}(\varepsilon^2);$$

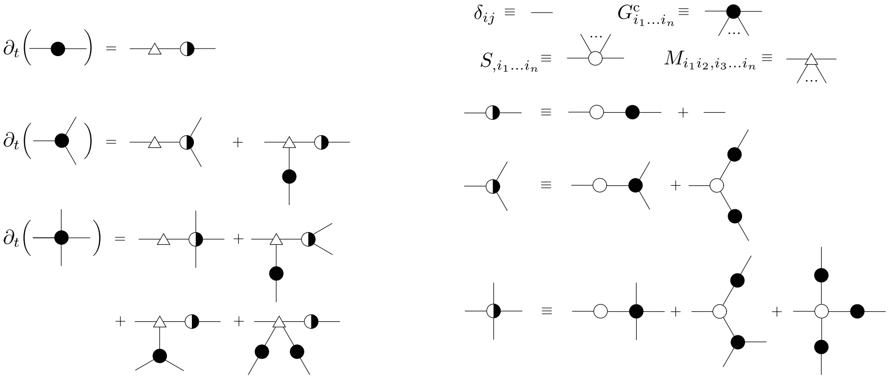
$$\vdots$$

$$\partial_t G_n^{\mathsf{c}} = -n\Gamma(G_n^{\mathsf{c}} - \bar{G}_n^{\mathsf{c}}[\psi, G, \ldots, G_{n-1}^{\mathsf{c}}]) + \mathcal{O}(\varepsilon^n);$$

  To leading order, the equations are iterative and “linear”.
- In hydrodynamics the small parameter is $(q/\Lambda)^3$, i.e., fluctuation wavelength $1/q \gg$ size of hydro cell $1/\Lambda$ (UV cutoff).

Diagrammatic representation

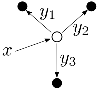
Systematically expand in ε and truncate at leading order:



- Leading order in ε $\Leftrightarrow$ tree diagrams.
- In higher-orders, loops describe feedback of fluctuations (e.g., long-time tails).

Generalizing Wigner transform

- Definition:



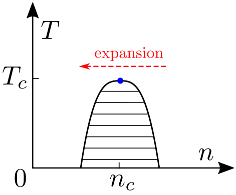
$$W_n(\boldsymbol{x};\boldsymbol{q}_1,\ldots,\boldsymbol{q}_n) \equiv \int d\boldsymbol{y}_1^3 \ldots \int d\boldsymbol{y}_n^3\, G_n\left(\boldsymbol{x}+\boldsymbol{y}_1,\ldots,\boldsymbol{x}+\boldsymbol{y}_n\right) \delta^{(3)}\left(\frac{\boldsymbol{y}_1+\ldots+\boldsymbol{y}_n}{n}\right) e^{-i(\boldsymbol{q}_1\cdot\boldsymbol{y}_1+\ldots+\boldsymbol{q}_n\cdot\boldsymbol{y}_n)};$$

$$G_n\left(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\right) = \int \frac{d\boldsymbol{q}_1^3}{(2\pi)^3} \ldots \int \frac{d\boldsymbol{q}_n^3}{(2\pi)^3}\, W_n(\boldsymbol{x},\boldsymbol{q}_1,\ldots,\boldsymbol{q}_n) \delta^{(3)}\left(\frac{\boldsymbol{q}_1+\ldots+\boldsymbol{q}_n}{2\pi}\right) e^{i(\boldsymbol{q}_1\cdot\boldsymbol{x}_1+\ldots+\boldsymbol{q}_n\cdot\boldsymbol{x}_n)}.$$

  - Properties similar to the usual ($n=2$) Wigner transform.
  - Takes advantage of the scale separation:
    long-scale x-dependence and short-scale $\boldsymbol{y}_n$-dependence.

Example: expansion through a critical region

An et al 2009.10742, PRL



- Two main features:
  - Lag, "memory".
  - Smaller Q – slower evolution. Conservation laws.
- Critical point signatures depend on the scale of fluctuations probed.