The challenge of $g-2$

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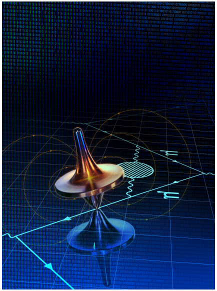

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Muon $g-2$ collab., PRL 126 (2021) 141801 (Featured in Physics) → FNAL '21
Aoyama et al., Phys. Rep. 887 (2020) 1-166 → WP '20
BMW collab., Nature 593 (2021) 51, online 7 April 2021 → BMWc '20
BMW collab., PRL 121 (2018) 022002 (Editors' Suggestion) → BMWc '17

Reference SM result vs experiment

SM contribution	$a_\mu^{\text{contrib.}} \times 10^{10}$	rel. err.	Ref.
QED [5 loops]	11658471.8931 ± 0.0104	[0.9 ppb]	[Aoyama '19, WP '20]
EW [2 loops]	15.36 ± 0.10	[0.7%]	[Gnendiger '15, WP '20]
HVP Tot. (R-ratio)	684.5 ± 4.0	[0.6%]	[WP '20]
HLbL Tot.	9.2 ± 1.8	[20%]	[WP '20]
SM	11659181.0 ± 4.3	[0.37 ppm]	[WP '20]

$$a_\mu|_{\text{exp.}} = 0.00116592061(41)$$
$$a_\mu|_{\text{ref.}} = 0.00116591810(43)$$
$$\text{diff.} = 0.00000000251(59)$$

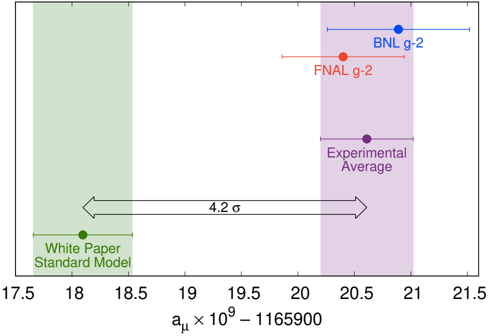
- Comparable errors but 4.2σ disagreement: probability $\lesssim 1/40\,000$

  $\Rightarrow$ evidence for BSM physics

- Particle physicists require probability $\lesssim 1/2\,000\,000$ to claim discovery (5σ)

Important to check most uncertain contribution (HVP) w/ fully independent methods

$\rightarrow$ *ab initio* calculations of contribution using **lattice quantum chromodynamics (QCD)**



Introduction to lattice QCD

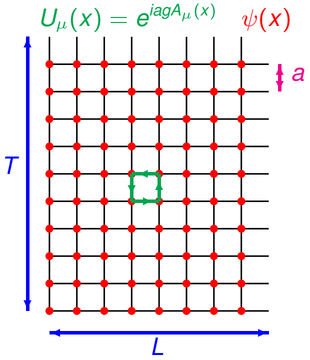
What is lattice QCD (LQCD)?

To describe low-energy, strong interaction phenomena w/ sub-% precision, QCD requires ≥ 128 numbers at every spacetime point
$\rightarrow \infty$ number of numbers in our continuous spacetime
$\rightarrow$ must temporarily "simplify" the theory to be able to calculate *(regularization)*
$\Rightarrow$ Lattice gauge theory $\longrightarrow$ mathematically sound definition of NP QCD:

- UV (& IR) cutoff $\rightarrow$ well defined path integral in Euclidean spacetime:

$$\begin{aligned} \langle O\rangle &= \int \mathcal{D}U\mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-S_G-\int \bar{\psi}D[M]\psi}\, O[U,\psi,\bar{\psi}] \\ &= \int \mathcal{D}U\, e^{-S_G}\, \mathrm{det}(D[M])\, O[U]_{\mathrm{Wick}} \end{aligned}$$

- $\mathcal{D}Ue^{-S_G}\,\mathrm{det}(D[M]) \geq 0$ & finite # of dofs $\rightarrow$ evaluate numerically using stochastic methods



LQCD is QCD when $m_q \rightarrow m_q^{\mathrm{ph}}$, $\Lambda_{\mathrm{QCD}} \rightarrow \Lambda_{\mathrm{QCD}}^{\mathrm{ph}}$, $a \rightarrow 0$ (after renormalization), $L \rightarrow \infty$ (and stats $\rightarrow \infty$)

HUGE conceptual and numerical ($O(10^{10})$ dofs) challenge

Our “accelerators”

Such computations require some of the world’s most powerful supercomputers

- copyright Photothèque CNRS/Cyril Frésillon

- 1 year on supercomputer
  $\sim$ 100 000 years on laptop

- In Germany, those of the Forschungszentrum Jülich, the Leibniz Supercomputing Centre (Munich), and the High Performance Computing Center (Stuttgart); in France, Turing and Jean Zay at the Institute for Development and Resources in Intensive Scientific Computing (IDRIS) of the CNRS, and Joliot-Curie at the Very Large Computing Centre (TGCC) of the CEA, by way of the French Large-scale Computing Infrastructure (GENCI).

Typical course of a LQCD calculation

Lattice QCD calculations generally proceed in 5 major steps:

(1) Gluon configurations, including vacuum fluctuations into $q = u$, d and s (and c) quarks, generated via HMC algorithm for variety of bare parameters and lattice sizes, chosen to allow (controlled) inter/extrapolation to physical point: $L \to \infty$ & $a \to 0$ while holding m_a and Λ_{QCD} fixed to their physical values
- $\to$ each set of configurations is an "ensemble"
- $\to$ basic ingredients of lattice calculations
- $\to$ can be used to obtain many different physical quantities

(2) Primary observables typically gauge-invariant products of quark propagators computed on each gauge configuration and averaged over these for each ensemble $\to$ Green's functions

(3) If necessary, renormalization constants for these Green's functions are computed, preferably nonperturbatively

(4) Physical observables extracted from the "poles" and "residues" of these Green's functions

(5) Renormalized observables, as functions of simulation parameters (or stand-ins), undergo thorough analyses, to determine their values at physical point w/ full stat. and syst. uncertainties

(1)-(3) require supercomputers for large lattices

(4)&(5) can be done on clusters

What is computed in LQCD and how?

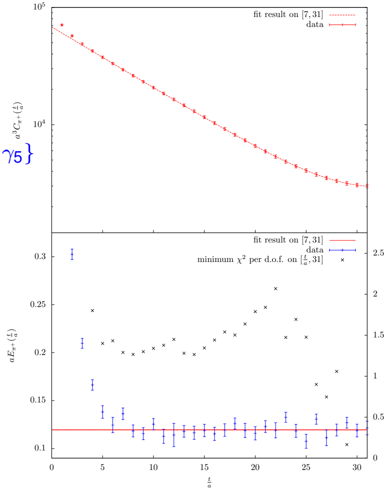
E.g.: compute & study time-dependence of an Euclidean 2-point function on a given ensemble

$$C(t,\vec{p}) \equiv a^3 \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \langle [\bar{d}\gamma_\mu\gamma_5 u](x)[\bar{u}\gamma_5 d](0)\rangle$$

$$= -\frac{a^3}{N}\sum_{U_1,\cdots,U_N}\sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}}\,\mathrm{tr}_{CD}\left\{U[x,0;U_i]\gamma_5 D[0,x;U_i]\gamma_\mu\gamma_5\right\}$$

$$\xrightarrow{(T-t)\to\infty} \sum_n \frac{\langle 0|[\bar{d}\gamma_\mu\gamma_5 u]|\pi_n(\vec{p})\rangle\langle\pi_n(\vec{p})|[\bar{u}\gamma_5 d]|0\rangle}{\langle\pi_n(\vec{p})|\pi_n(\vec{p})\rangle} e^{-E_n t}$$

$$\xrightarrow{t,\,(T-t)\to\infty} \frac{\langle 0|[\bar{d}\gamma_\mu\gamma_5 u]|\pi(\vec{p})\rangle\langle\pi(\vec{p})|[\bar{u}\gamma_5 d]|0\rangle}{\langle\pi(\vec{p})|\pi(\vec{p})\rangle} e^{-E_\pi t}$$



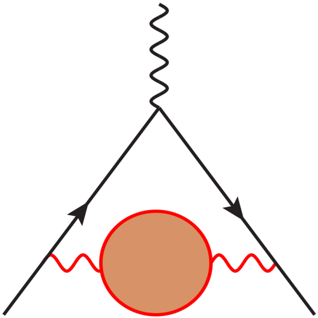
From fits, extract $E_\pi = \sqrt{M_\pi^2 + \vec{p}^2}$ and $\langle 0|[\bar{d}\gamma_\mu\gamma_5 u]|\pi(\vec{p})\rangle = p_\mu f_\pi$ w/ $p_\mu = (E_\pi, \vec{p})$

$$aM_{\mathrm{eff}}(t/a) \equiv \ln\frac{C(t+1)}{C(t)}$$

Repeat on different ensembles and take physical limit

Also 3 and 4-point functions, but statistical noise typically increases with number of points

Lattice QCD calculation of $a_\mu^{\text{LO-HVP}}$



All quantities related to a_μ will be given in units of 10^{-10} unless stated otherwise

HVP from LQCD: introduction

Consider in Euclidean spacetime, i.e. spacelike $q^2 = -Q^2 \leq 0$ [Blum '02]

$$\Pi_{\mu\nu}(Q) =$$

$$= \int d^4x\, e^{iQ\cdot x} \langle J_\mu(x) J_\nu(0)\rangle$$

$$= \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\right) \Pi(Q^2)$$

w/ $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \cdots$

Then [Lautrup et al '69, Blum '02]

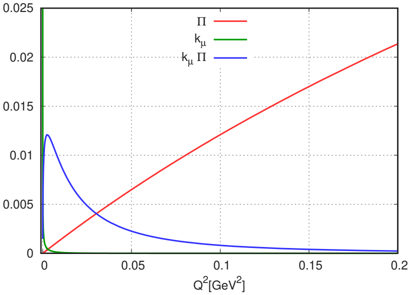
$$a_\ell^{\text{LO-HVP}} = \alpha^2 \int_0^\infty \frac{dQ^2}{m_\ell^2} k(Q^2/m_\ell^2) \hat{\Pi}(Q^2)$$

w/ $\hat{\Pi}(Q^2) \equiv \left[\Pi(Q^2) - \Pi(0)\right]$ and

$$k(r) = \left[r + 2 - \sqrt{r(r+4)}\right]^2 / \sqrt{r(r+4)}$$

Integrand peaked for $Q \sim (m_\ell/2) \sim 50\,\text{MeV}$ for μ

$\Rightarrow$ clearly in QCD's nonperturbative regime



$(k_\mu(Q^2) = (\pi/m_\mu)^2 k(Q^2))$

Low-Q^2 challenges in finite volume (FV)

A. In L^4, $Q_\mu \Pi_{\mu\nu}(Q) = 0$ does not imply $\Pi_{\mu\nu}(Q=0) = 0$

$$\Pi_{\mu\nu}(Q=0) = \int_\Omega d^4x \langle J_\mu(x) J_\nu(0)\rangle = \int_\Omega d^4x \partial_\rho [x_\mu \langle J_\rho(x) J_\nu(0)\rangle]$$

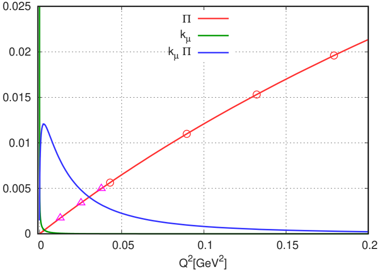
$$\int_{\partial\Omega} d^3x_\rho [x_\mu \langle J_\rho(x) J_\nu(0)\rangle] \propto L^4 \exp(-EL/2)$$

$\Rightarrow$ as $Q_\mu \to 0$, $\Pi(Q^2) = \Pi_{\mu\nu}(Q)/(Q_\mu Q_\nu - Q^2\delta_{\mu\nu})$ receives $1/Q^2$ enhanced FV effect

B. Particularly problematic, as need $\Pi(0)$ for renormalization

C. Need $\hat{\Pi}(Q^2)$ interpolation because in $T \times L^3$, w/ periodic BCs, have sparse momenta around $\frac{m_\mu}{2} \sim 50\,\mathrm{MeV}$

- $Q = (\frac{2\pi}{T} n_0, \frac{2\pi}{L}\vec{n})$ w/ $n \in \mathbb{Z}^4$
- $|Q| \simeq (110, 205)\,\mathrm{MeV}$ for $(n_0, |\vec{n}|) = (1,0)$ & $(n_0, |\vec{n}|) = (0,1)$ w/ $(T, L) \simeq (11, 6)\,\mathrm{fm}$



$(k_\mu(Q^2) = (\pi/m_\mu)^2 k(Q^2))$

Dealing with low-Q^2 problems: ad A, B & C

- Compute on $T \times L^3$ lattice in $N_f = 2+1+1$ QCD

$$C^{\text{iso}}_{TL}(t) = \frac{a^3}{3}\sum_{i=1}^{3}\sum_{\vec{x}} \langle J_i(x) J_i(0)\rangle$$

- Decompose ($C^{I=1}_{TL} = \frac{9}{10} C^{ud}_{TL}$)

$$C^{\text{iso}}_{TL}(t) = C^{ud}_{TL}(t) + C^{s}_{TL}(t) + C^{c}_{TL}(t) + C^{\text{disc}}_{TL}(t) = C^{I=1}_{TL}(t) + C^{I=0}_{TL}(t)$$

- Define (Bernecker et al '11, BMWc '13, Lehner '14, ...) (ad A, B) [see also Charles et al '17]

$$\begin{aligned}\hat{\Pi}^f_{TL}(Q^2) &\equiv \Pi^f_{TL}(Q^2) - \Pi^f_{TL}(0)\\ &= \frac{1}{3}\sum_{i=1}^{3}\frac{\Pi^f_{ii,TL}(0) - \Pi^f_{ii,TL}(Q)}{Q^2} - \Pi^f_{TL}(0)\\ &= a\sum_{t=0}^{T-a} \mathrm{Re}\left[\frac{e^{iQt}-1}{Q^2} + \frac{t^2}{2}\right] \mathrm{Re}\, C^f_{TL}(t)\end{aligned}$$

- Consider also for $Q \in \mathbb{R} \neq n\frac{2\pi}{T}$, $n \in \mathbb{Z}$ (RBC/UKQCD '15, ...) (ad C)

  $\rightarrow$ gives $a_\mu^{\text{LO-HVP}}$ up to exponentially suppressed FV corrections

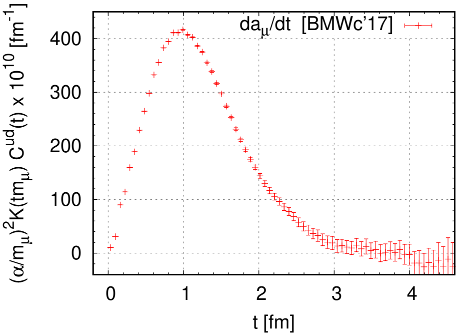
Our lattice definition of $a_{\ell,f}^{\text{LO-HVP}}$

Combining everything, get $a_{\ell,f}^{\text{LO-HVP}}$ from $C_{TL}^f(t)$ [Bernecker et al '11, BMWc '13, Feng et al '13, Lehner '14, ...]

$$a_{\ell,f}^{\text{LO-HVP}}(Q^2 \leq Q^2_{\text{max}}) = \lim_{a\to 0,\, L\to\infty,\, T\to\infty} \alpha^2 \left(\frac{a}{m_\ell^2}\right) {\sum_{t=0}^{T/2}}' K(tm_\ell, Q^2_{\text{max}}/m_\ell^2)\ \mathrm{Re}C_{TL}^f(t)$$

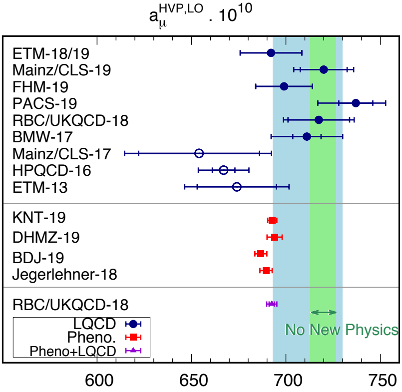
where

$$K(\tau, r_{\text{max}}) = \int_0^{r_{\text{max}}} dr\, k(r) \left(\tau^2 - \frac{4}{r}\sin^2\frac{\tau\sqrt{r}}{2}\right)$$



(144×96^3, $a \sim 0.064$ fm, $M_\pi \sim 135$ MeV)

Situation before BMWc'20



(WP'20)

R-ratio: $a_\mu^{\text{LO-HVP}} = (693.1 \pm 4.0) \times 10^{-10}$ [0.6%]

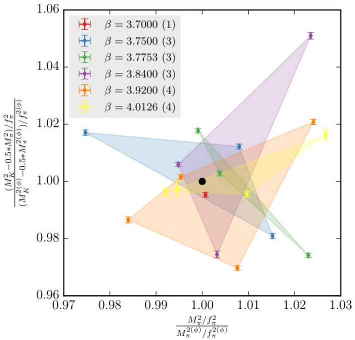
BMWc'17: $a_\mu^{\text{LO-HVP}} = (711.1 \pm 18.9) \times 10^{-10}$ [2.7%]

$\Rightarrow$ to be competitive w/ R-ratio, must reduce total uncertainty on BMWc'17 by ~ 4

Simulation details

31 high-statistics simulations w/ N_f=2+1+1 flavors of 4-stout staggered quarks:

- 6 a's: 0.134 → 0.064 fm
- Bracketing physical m_{ud}, m_s, m_c
- $L = 6.1 \div 6.6$ fm, $T = 8.6 \div 11.3$ fm
- Conserved EM current



+4 dedicated, N_f = 2+1, 4-HEX, FV simulations w/ $a = 0.112$ fm and $L = 6.3$ and 10.7 fm bracketing physical m_{ud}, m_s

β	a [fm]	$T \times L$	#conf
3.7000	0.1315	64 × 48	904
3.7500	0.1191	96 × 56	2072
3.7753	0.1116	84 × 56	1907
3.8400	0.0952	96 × 64	3139
3.9200	0.0787	128 × 80	4296
4.0126	0.0640	144 × 96	6980

For sea-quark QED corrections

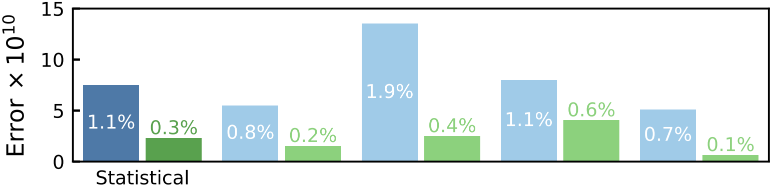
β	a [fm]	$T \times L$	#conf
3.7000	0.1315	48 × 24	716
		64 × 48	300
3.7753	0.1116	56 × 28	887
3.8400	0.0952	64 × 32	4253

- State-of-the-art techniques:
  - EigCG [Strathopoulos et al '08]
  - Low mode averaging [Neff et al '01, Giusti et al '04, ...]
  - All mode averaging [Blum et al '13]
  - Solver truncation [Bali et al '09]

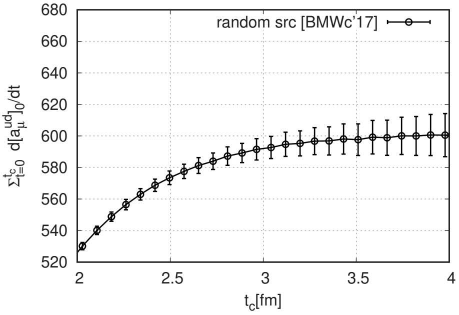
⇒ Over 25,000 gauge configurations

⇒ 10's of millions of measurements

Key improvements: statistical noise reduction



Exponentially increasing noise-to-signal ratio in $C_{TL}^{ud}(t)$ (and $C_{TL}^{\rm disc}(t)$): N/S $\sim \exp\{(M_\rho - M_\pi)t\}$



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- Above $t_c \gtrsim 3$ fm, $C_{TL}^{ud}(t)$ adds mostly noise: stat. err. $\gtrsim 0.7\%$ [BMWc'17]
- Stop sum where rigorous upper/lower bounds meet ($t_c \simeq 3$ fm) and take average (similarly for $C_{TL}^{\rm disc}(t)$) [BMWc '17]

$$0 \le C_L^{ud}(t) \le C_L^{ud}(t_c)\, e^{-E_{2\pi}(t-t_c)}$$

- BMWc'20–LMA: use exact (all-to-all) quark propagators in IR and stochastic in UV [Neff et al '01, Giusti et al '04]
- BMWc'20–increase statistics to $> 25,000$ gauge configs & many 10^7 measurements

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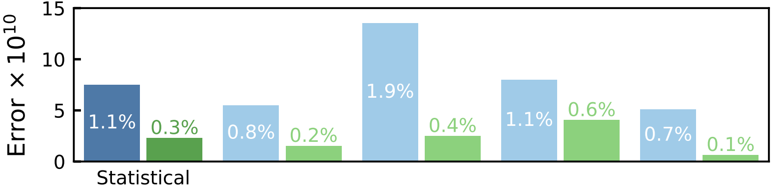

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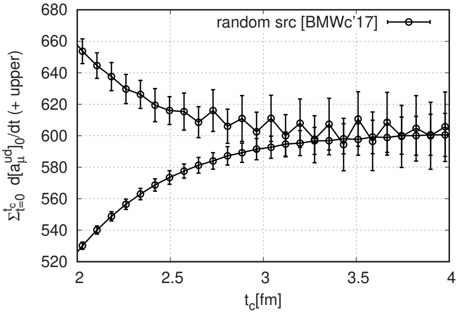

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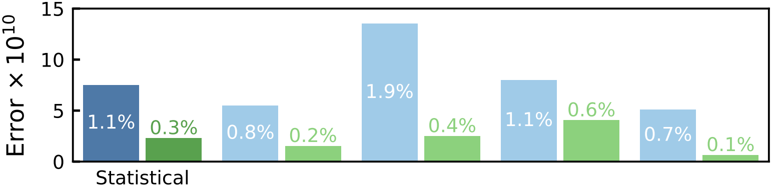

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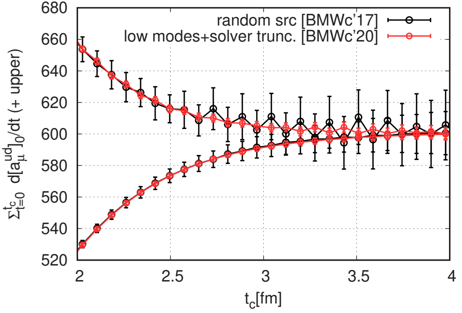

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- Stop sum where rigorous upper/lower bounds meet ($t_c \simeq 4$ fm) and take average (similarly for $C^{\rm disc}_{TL}(t)$) [BMWc '20]

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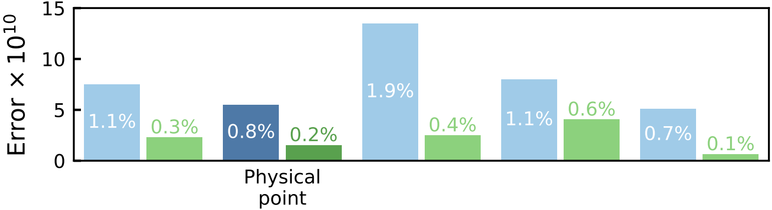
- BMWc'20–increase statistics to $> 25,000$ gauge configs & many 10^7 measurements

Key improvements: tuning of QCD parameters

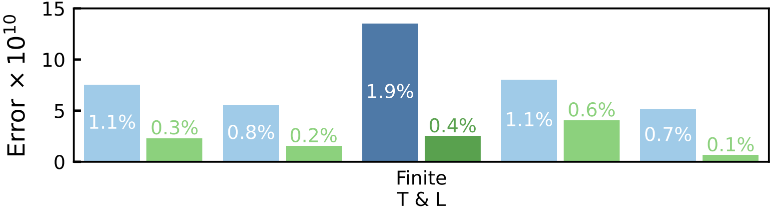


Must tune parameters of QCD very precisely: m_u, m_d, m_s, m_c & overall mass scale

Solve w/:

- Permil determination of overall QCD scale: set w/ Ω^- baryon mass computed w/ 0.2% uncertainty
- Quark masses set using $M^2_{\pi^0}$, $M^2_{ss} = M^2_{K^+} + M^2_{K^0} - M^2_{\pi^+}$, $\Delta M^2_K = M^2_{K^0} - M^2_{K^+}$, $m_c/m_s = 11.85$ (Davies et al, '10) computed w/ commensurate precision

Key improvements: remove finite spacetime distortions

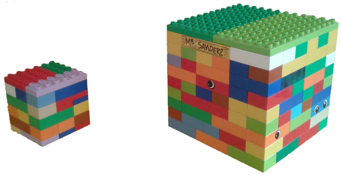


Even on "large" lattices ($L \gtrsim 6\,\mathrm{fm}$, $T \gtrsim 9\,\mathrm{fm}$), early pen-and-paper estimate [Aubin et al '16] suggested that exponentially suppressed finite-volume distortions are still $O(2\%)$

Solve by:

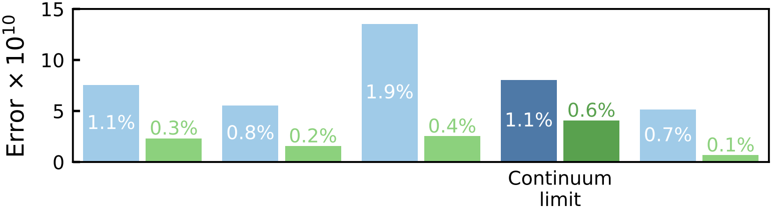
- Finding a way to perform dedicated supercomputer simulations to calculate effect between above and much larger $L = T = 11\,\mathrm{fm}$ volume directly in QCD, i.e. "big" − "ref"
- Computing remnant $\sim 0.1\%$ effect of "big" volume w/ EFTs that correctly predict "big" − "ref"

"ref" →

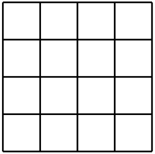


← "big"

Key improvements: controlled continuum limit



Our world corresponds to spacetime w/ lattice spacing $a \to 0$



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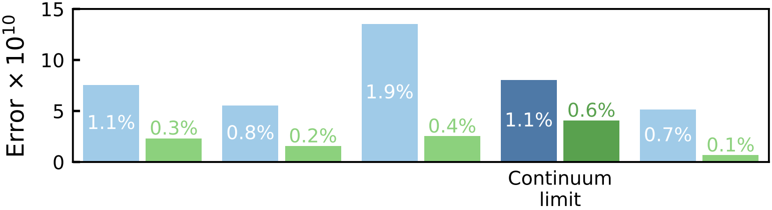

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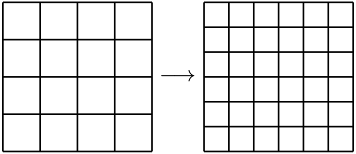

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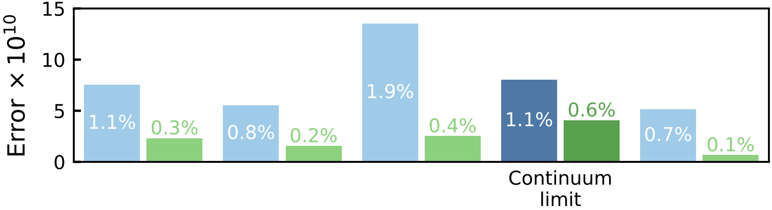

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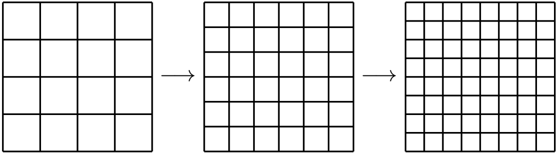

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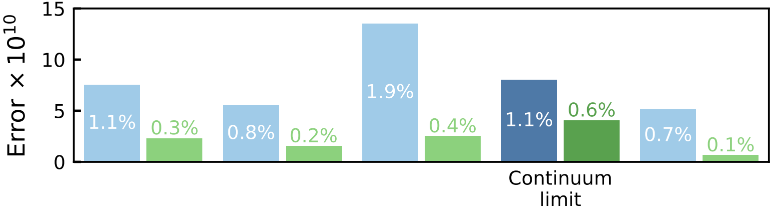

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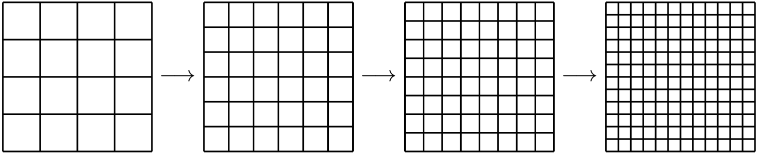
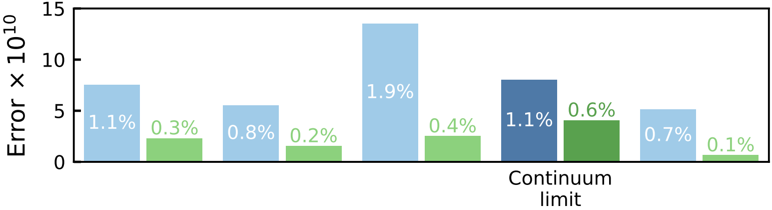
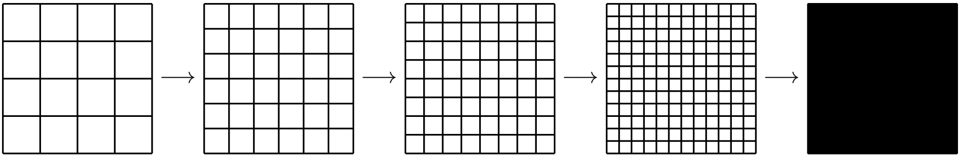


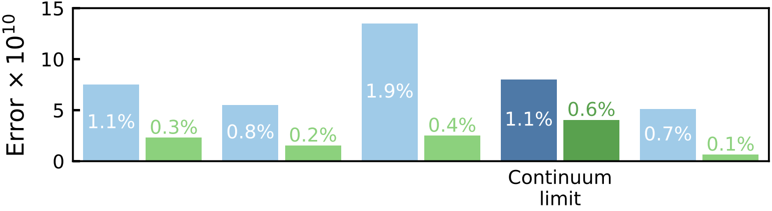
Key improvements: controlled continuum limit



Our world corresponds to spacetime w/ lattice spacing $a \to 0$



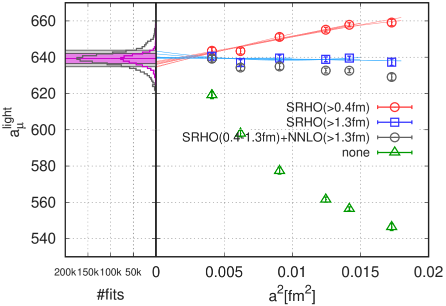
Key improvements: controlled continuum limit



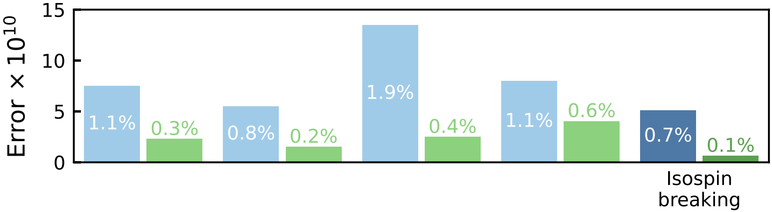
Our world corresponds to spacetime w/ lattice spacing $a \to 0$

Control $a \to 0$ extrapolation of results by:

- Performing all calculations on lattices w/ 6 values of a in range $0.134\ \mathrm{fm} \to 0.064\ \mathrm{fm}$
- Reducing statistical error at smallest a from 1.9% to 0.3% !
- Improving approach to continuum limit w/ pheno. models for QCD [Sakurai '60, Bijnens et al '99, Jegerlehner et al '11, Chakraborty et al '17, BMWc '20] shown to reproduce distortions observed at $a>0$
- Extrapolate results to $a=0$ using theory as guide



Key improvements: QED and $m_u \neq m_d$ corrections



For subpercent accuracy, must include small effects from electromagnetism and due to fact that masses of *u* and *d* quarks are not quite equal

- Effects are proportional to powers of $\alpha = \frac{e^2}{4\pi} \sim 0.01$ and $\frac{m_d - m_u}{(M_p/3)} \sim 0.01$

$\Rightarrow$ for SM calculation at permil accuracy sufficient to take into account contributions proportional to first power of α or $\frac{m_d - m_u}{(M_p/3)}$

- We include *all* such contributions for *all* calculated quantities needed in calculation

Including isospin breaking on the lattice

$$S_{\rm QCD+QED} = S^{\rm iso}_{\rm QCD} + \frac{1}{4}\int F^2 + \frac{1}{2}\delta m\int(\bar{d}d - \bar{u}u) + ie\int A_\mu J_\mu, \qquad J_\mu = \bar{q}Q\gamma_\mu q, \qquad \delta m = m_d - m_u$$

- Separation into isospin limit results and corrections requires an unambiguous definition of this limit (scheme and scale)
- Must be included not only in calculation of $\langle J_\mu J_\nu\rangle$ correlator **BUT ALSO** of all quantities used to fix quark masses and QCD scale

(1) operator insertion method [RM123 '12, '13, ...]

$$\langle \mathcal{O}\rangle_{\rm QCD+QED} = \langle \mathcal{O}_{\rm Wick}\rangle^{\rm iso}_{G_\mu} - \frac{\delta m}{2}\langle[\mathcal{O}\int(\bar{d}d - \bar{u}u)]_{\rm Wick}\rangle^{\rm iso}_{G_\mu} - \frac{e^2}{2}\langle[\mathcal{O}\int_{xy} J_\mu(x)D_{\mu\nu}(x-y)J_\nu(y)]_{\rm Wick}\rangle^{\rm iso}_{G_\mu}$$
$$+e^2\langle\langle\left[\mathcal{O}\partial_e\frac{\det D[G_\mu, eA_\mu]}{\det D[G_\mu,0]}|_{e=0}\int_x J_\mu(x)A_\mu(x) - \frac{1}{2}\mathcal{O}\partial_e^2\frac{\det D[G_\mu, eA_\mu]}{\det D[G_\mu,0]}|_{e=0}\right]_{\rm Wick}\rangle_{A_\mu}\rangle^{\rm iso}_{G_\mu}$$

(2) direct method [Eichten et al '97, BMWc '14, ...]

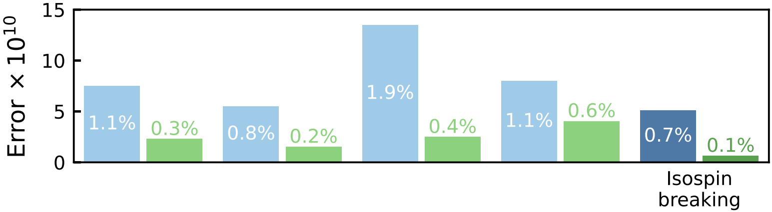
Include $m_u \neq m_d$ and QED directly in calculation of observables and generation of gauge configurations

(3) combinations of (1) & (2) [BMWc '20]

We include ALL $O(e^2)$ and $O(\delta m)$ effects

For valence e^2 effects use easier (2), and for δm and e^2 sea effects, (1)

Key improvements: QED and $m_u \neq m_d$ corrections



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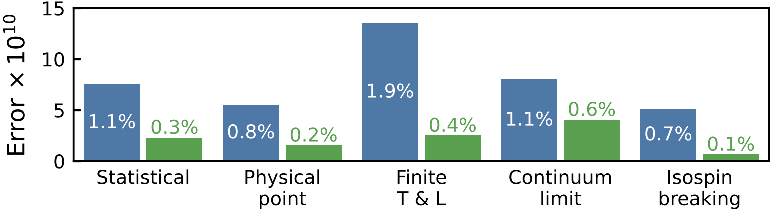
Robust determination of uncertainties

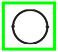


Thorough and robust determination of statistical and systematic uncertainties

- Stat. err.: resampling methods
- Syst. err.: extended frequentist approach [BMWc '08, '14]
  - Hundreds of thousands of different analyses of correlation functions
  - Weighted by AIC weight for physical point inter/extrapolation and flat for other variations
  - Use median of distribution for central values & $16 \div 84\%$ confidence interval to get total error

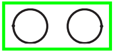
(Nature paper has 95 pp "Supplementary Information" detailing methods)

Isospin symmetric

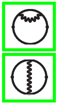

connected light
633.7(2.1)(4.2)

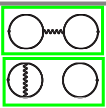

connected strange
53.393(89)(68)


connected charm
14.6(0)(1)


disconnected
-13.36(1.18)(1.36)

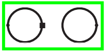
QED isospin-breaking: valence


connected -1.23(40)(31)

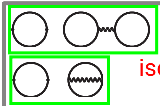

disconnected -0.55(15)(10)

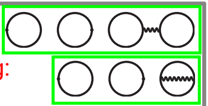
Strong isospin-breaking


connected
6.60(63)(53)


disconnected
-4.67(54)(69)

QED isospin-breaking: sea


connected 0.37(21)(24)

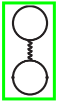

disconnected -0.040(33)(21)

Etc.

bottom; higher order; perturbative

0.11(4)

QED isospin-breaking: mixed


connected -0.0093(86)(95)


disconnected 0.011(24)(14)

Finite-size effects

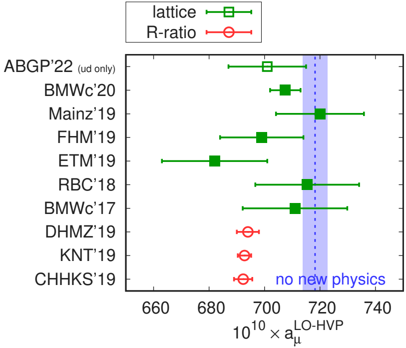
isospin-symmetric
18.7(2.5)

isospin-breaking
0.0(0.1)

$$10^{10}\times a_\mu^{\text{LO-HVP}} = 707.5(2.3)_{\text{stat}}(5.0)_{\text{sys}}[5.5]_{\text{tot}}$$

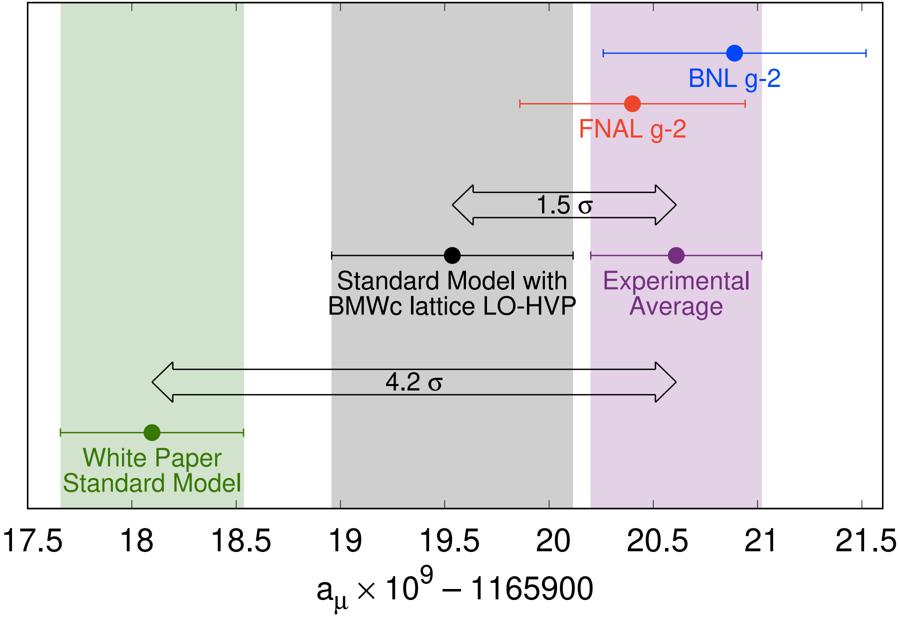
Comparison and outlook

Comparison



- Consistent with other lattice results
- Total uncertainty is divided by $3 \div 4$ ...
- ... and comparable to R-ratio and $(g-2)$ experiment
- 2.1σ larger than R-ratio average value [WP '20]
- Further confirmed by very recent ABGS'22 whose uncertainties, however, make it also consistent w/ R-ratio determination
- Consistent w/ a_μ measurement @ 1.5σ level ("no new physics" scenario) !

Fermilab plot, BMWc version



Useful window

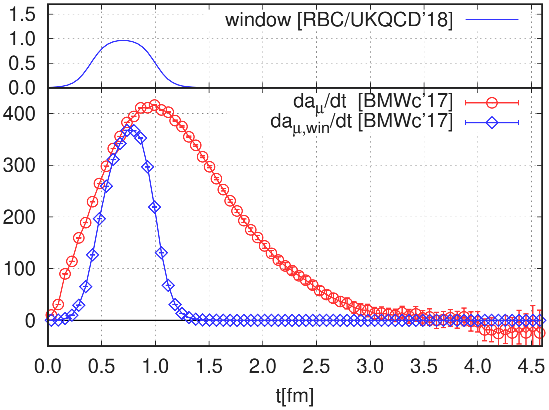
Window functions:

$$\Theta(t;t_0,\Delta) \equiv \frac{1}{2}\left[1+\tanh\left(\frac{t-t_0}{\Delta}\right)\right]$$

$$W(t;t_0,t_1,\Delta) \equiv \Theta(t;t_0,\Delta)-\Theta(t;t_1,\Delta)$$

$$W_{\rm slide}(t;t_0,\Delta) \equiv W(t;t_0,t_0{+}0.5\,\mathrm{fm},\Delta)$$

Particularly clean:
$W(t;0.4\,\mathrm{fm},1.0\,\mathrm{fm},0.15\,\mathrm{fm})$



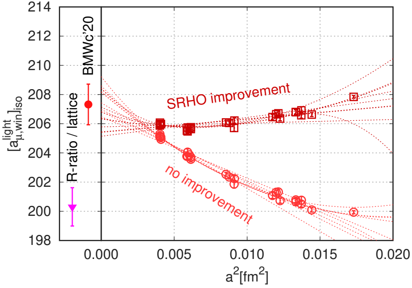
$(144 \times 96^3,\ a \sim 0.064\ \mathrm{fm},\ M_\pi \sim 135\ \mathrm{MeV})$

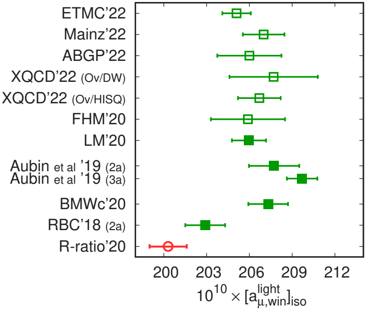
Window results

- Less challenging than full $a_\mu^{\text{LO-HVP}}$
  - much better signal/noise
    $\to$ stat. err. $\leq 0.2\%$
  - much smaller FV effects $\lesssim 0.3\%$
  - much smaller discretization effects (long & short distance) $\lesssim 2.7\%$ for $a \leq 0.1\,\text{fm}$
    $\to$ include $a \to 0$ w/ and w/out taste improvement
    $\to$ very conservative systematics
  - tot. err. $\sim 0.7\%$ of which 88% comes from $a \to 0$

$\to$ other LQCD groups have comparable errors

- 3.7σ tension w/ R-ratio
- 7.0 out of 14.4 lattice vs R-ratio excess in $10^{10} \times a_\mu^{\text{LO-HVP}}$
- Schwinger Fest, 14-17/6/22: Mainz'22 & ETMC confirm BMWc'20 result for $a_{\mu,\text{win}}^{\text{LO-HVP}}$ using different fermion discretization (Wilson) and fine lattices





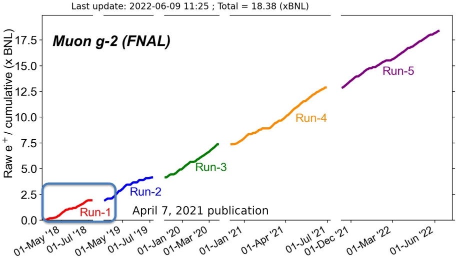
Conclusions and outlook

- a_μ is measured to 0.35 ppm and predicted in SM to 0.37 ppm
- BMWc'20's lattice QCD calculation of $a_\mu^{\text{LO-HVP}}$ reaches precision comparable to reference $e^+e^- \to$ hadrons approach for first time
- While reference SM prediction [WP'20] gives $a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 25.1(5.9) \times 10^{-10}$, i.e. a 4.2σ indication of new physics, . . .
- . . . lattice QCD calculation reduces this difference to 1.5σ, $a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 10.7(1.7) \times 10^{-10}$ . . .
- . . . at expense of 2.1σ tension w/ $a_\mu^{\text{LO-HVP}}$ & 3.7σ tension w/ $a_{\mu,\text{win}}^{\text{LO-HVP}}$ from $e^+e^- \to$ hadrons
- News: confirmation of $a_{\mu,\text{win}}^{\text{LO-HVP}}$ tension at Schwinger Fest '22
- Still awaiting results from RBC/UKQCD & FNAL/HPQCD/MILC
- Neverthelss likely that this $> 3\sigma$ tension w/ R-ratio remains
- Of course, need confirmation of high lattice value for much more challenging $a_\mu^{\text{LO-HVP}}$

Conclusions and outlook

- Upcoming experimental progress:
  - Results of Run 2/3 expected early 2023 w/ $\delta_{\text{tot}} a_\mu \sim 0.23\,\text{ppm}$ including significantly reduced $0.10\,\text{ppm}$ systematics
  - Ongoing Run 5 should allow to reach BNL $\times 19$
  - Run 6 w/ μ^- in '22-'23

  → WA experimental error reduced by 1.5 in '23 and 2.5 around '26

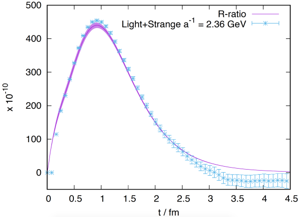


- Must reduce error on HLbL by $1.5 \div 2 \ldots$
- … & lattice HVP error by ~ 4!
- Must also reduce share of *systematic* error on HVP
- The whole picture can still change!

Conclusions and outlook

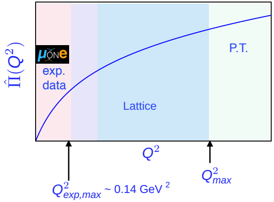
- If lattice HVP fully confirmed by other groups, must understand source of disagreement with R-ratio approach
- If disagreement can be fixed, combine LQCD and phenomenology to improve overall uncertainty

  [RBC/UKQCD '18]



[RBC/UKQCD '18]

- Important to pursue $e^+e^- \to$ hadrons measurements [BaBar, CMD-3, BES III, Belle II, ...]
- $\mu e \to \mu e$ experiment MUonE very important for experimental crosscheck and complementarity w/ LQCD



[Marinkovic et al '19]

- Important to pursue J-PARC $g_\mu - 2$ and pursue a_e experiments

Many thanks to my collaborators

Borsanyi, Fodor, Guenther, Hoelbling, Katz, LL, Lippert, Miura, Szabo, Parato, Stokes, Toth, Torok, Varnhorst [Budapest-Marseille-Wuppertal collaboration], Nature 593 (2021) 51 $\rightarrow$ BMWc '20