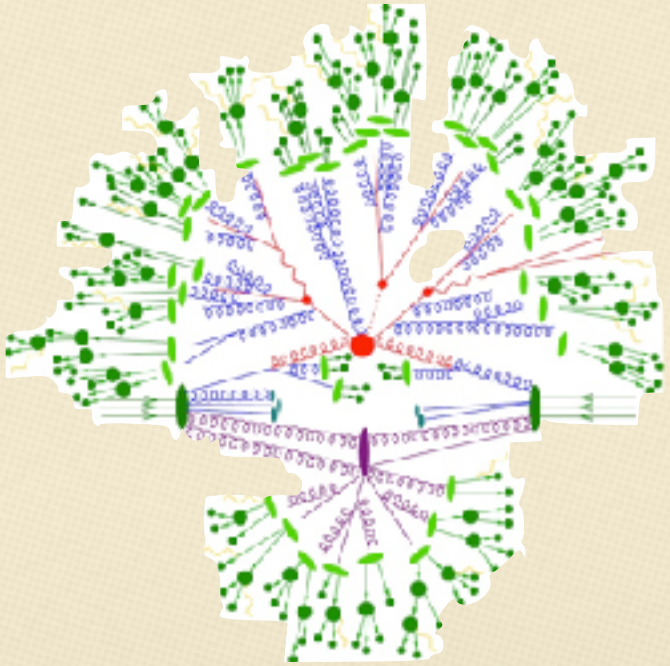
$$\frac{1}{4}\mathrm{Tr}[G^2] - \bar{\psi}(\not{D} - m)\psi$$ ☞



LHC and perturbative QCD

2022 Erice School of Subnuclear Physics

Gravity and matter in the subnuclear world

Eric Laenen

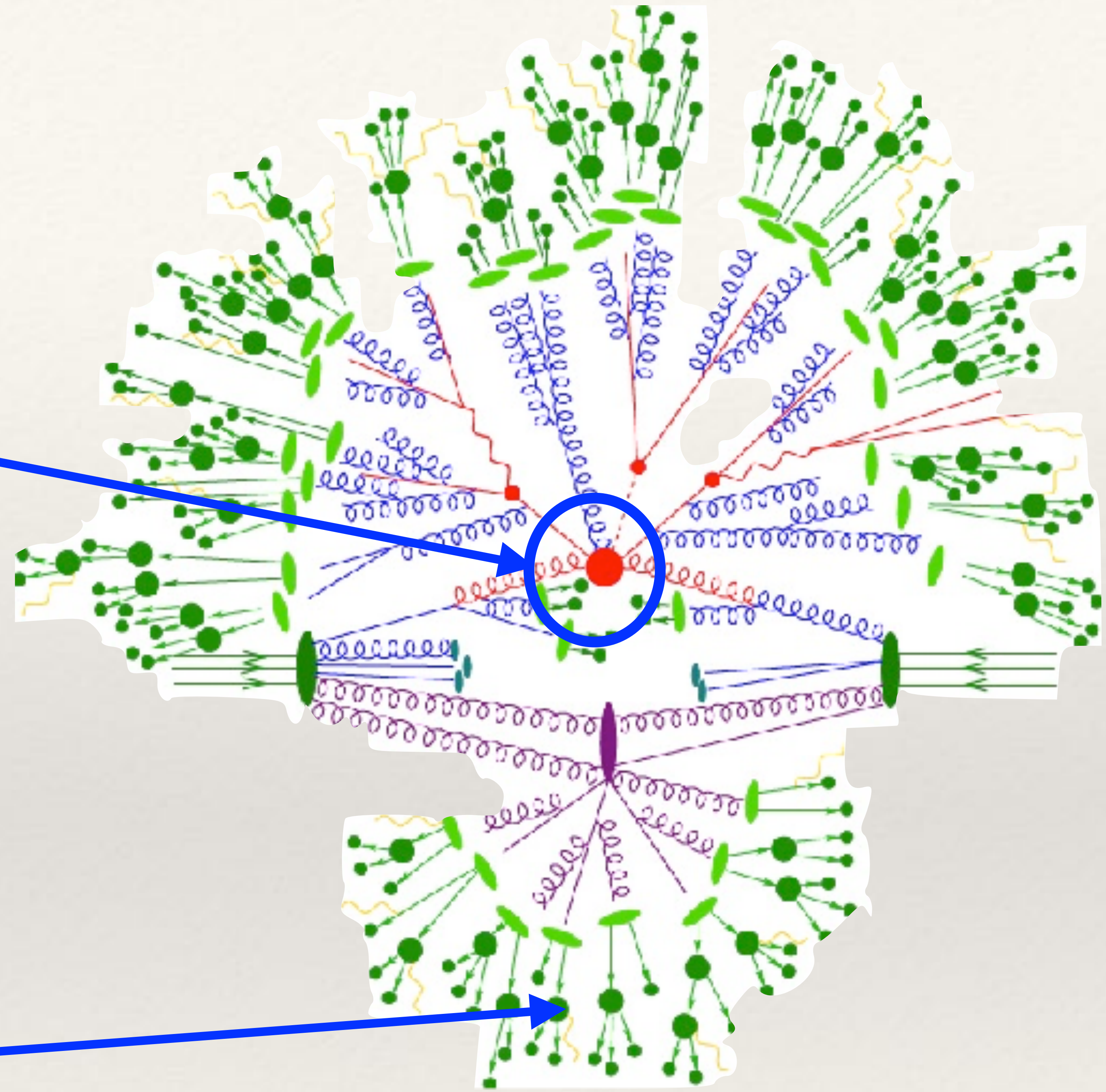
Universiteit van Amsterdam

Nikhef

Utrecht University

Where are we?

- PDFs ✓
- Partonic processes
- Parton showers ✓
- ~~Hadronization~~

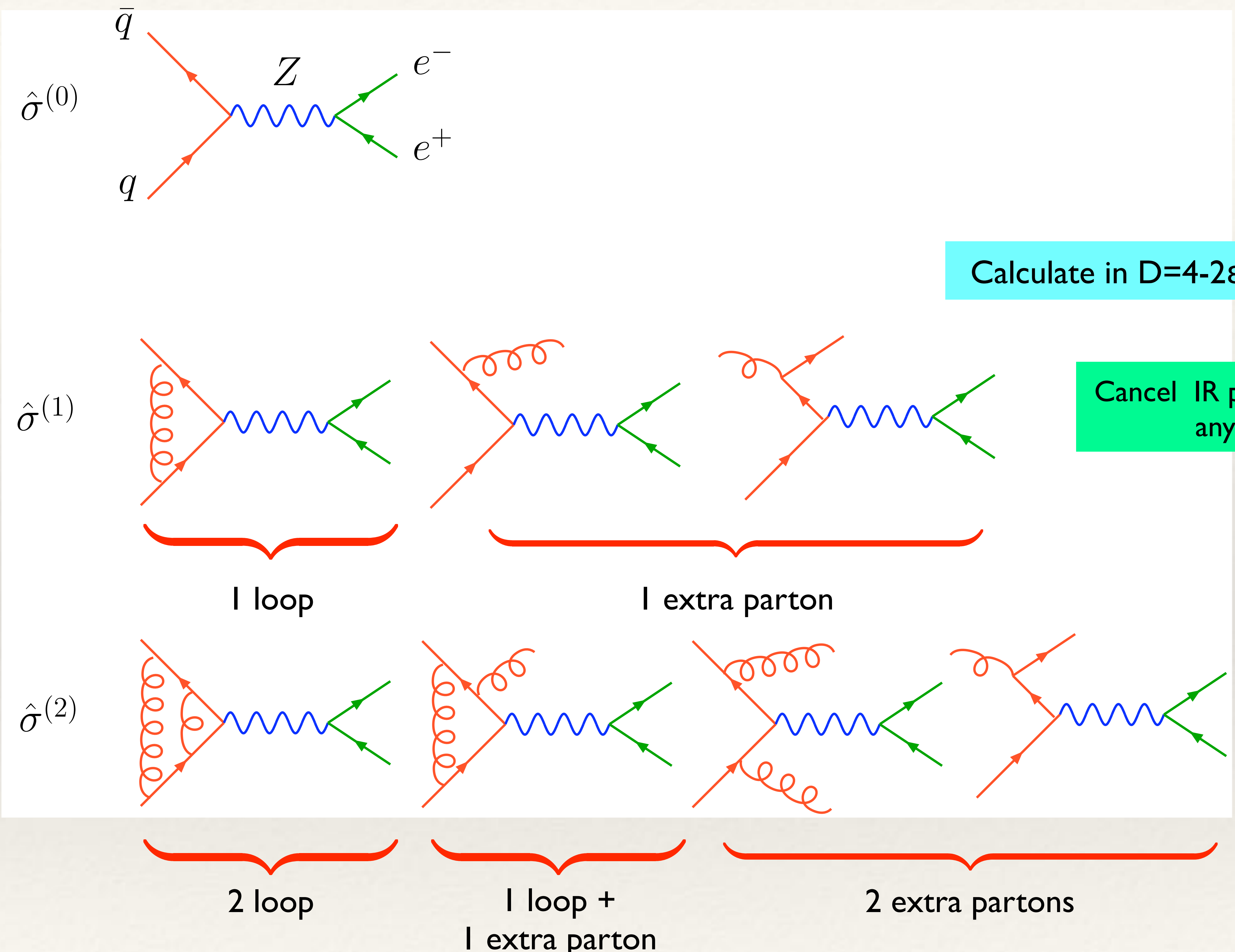


Recall: LO and higher order amplitudes

LO

NLO

NNLO



Calculate in D=4-2ε dimensions

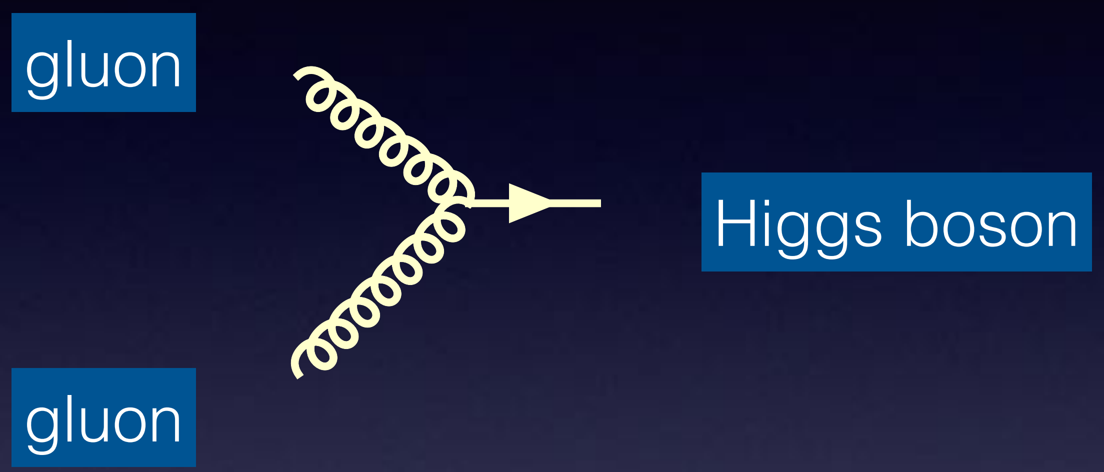
Cancel IR poles 1/ε² before anything else

Cancel IR poles 1/ε⁴ etc before anything else; hard!

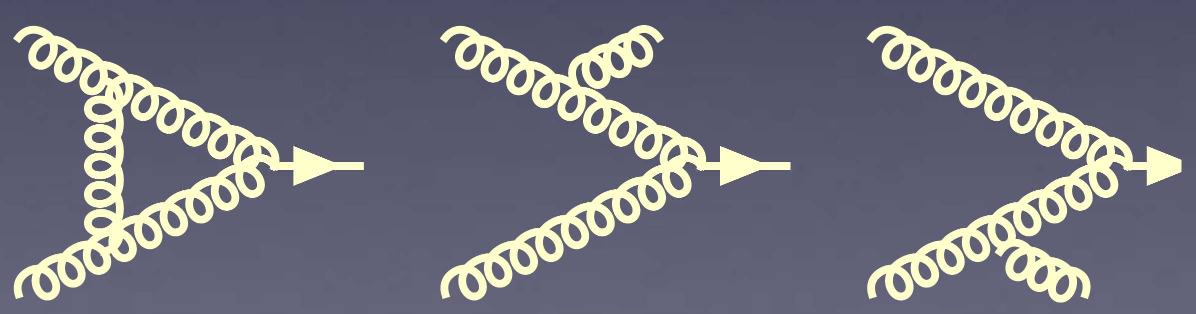
After G. Zanderighi

Higgs production cross section in perturbation theory

Leading order prediction based on 1 simple diagram:

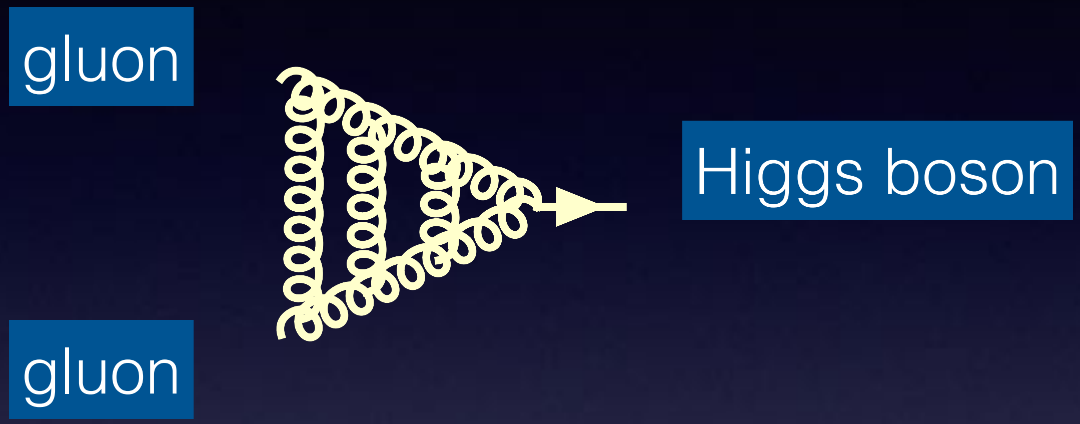


Next-to-leading order (NLO) prediction needs more:



NNNLO Higgs production

- 7 million (3-loop) Feynman diagrams

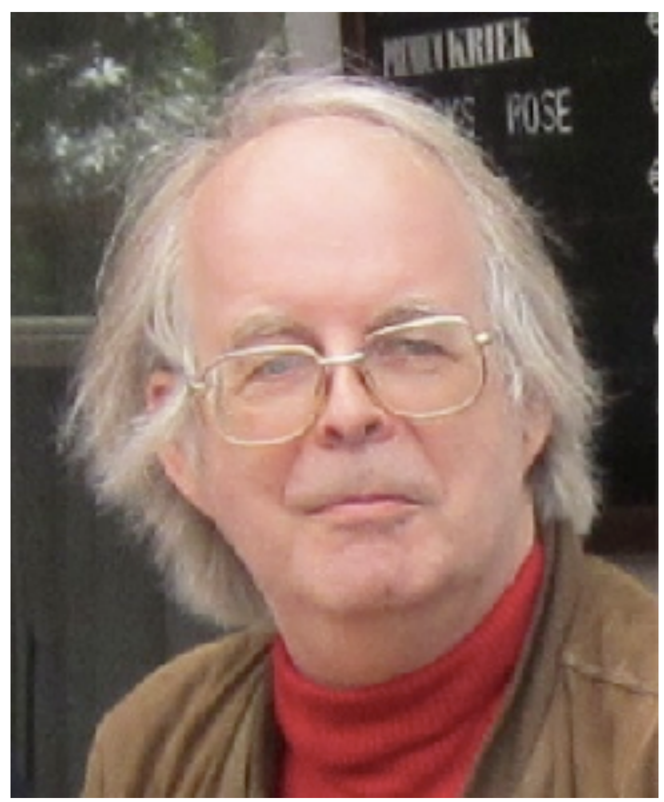


- Each needs powerful algebra, and requires advanced mathematical methods

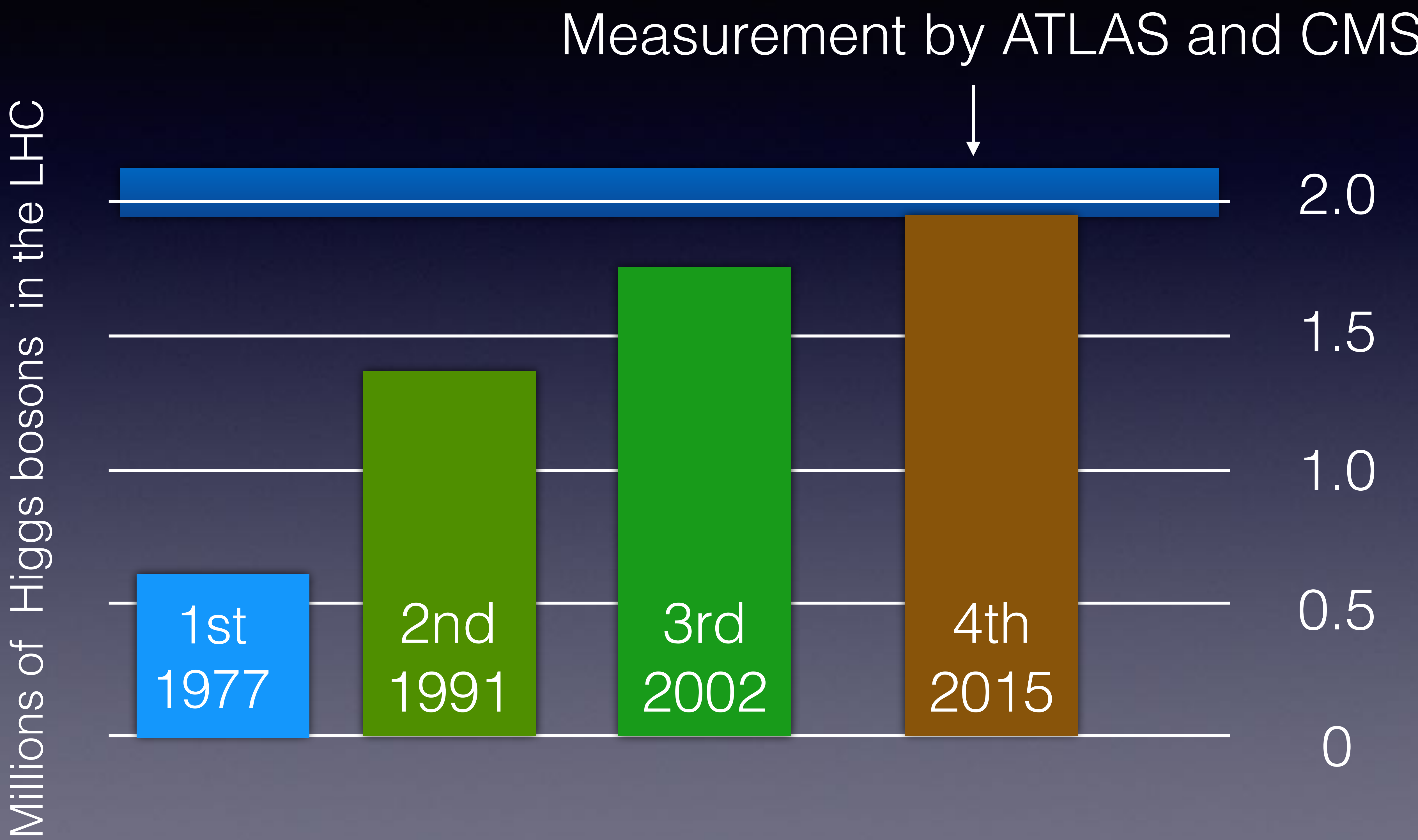
- Total: 2 months on 25000 computers in parallel

FORM computer algebra program

- Extraordinarily powerful, vital for collider physics calculations
  - developed by Jos Vermaseren@Nikhef over 30 years, written in C
  - Successor to Veltman’s Schoonschip programme
  - Now open source, various contributors



Perturbative orders Higgs production cross section



Modern methods for loop integrals

Feynman parameters

- Standard one-loop master formula in DimReg

$$\int \frac{d^n l}{(2\pi)^n} \frac{1}{[l^2 - M^2 + i\varepsilon]^s} = \frac{i\pi^{n/2}}{(2\pi)^n} (-)^s \frac{\Gamma(s - n/2)}{\Gamma(s)} (M^2 - i\varepsilon)^{n/2-s}$$

  - Notice the Gamma functions..

- To use it, need an expression with 1 denominator. Great trick by Feynman:

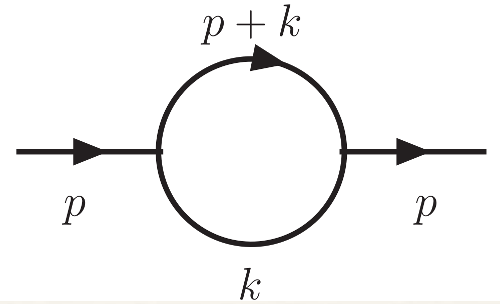
$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{[xA + (1-x)B]^2}$$

  - Full generalization

$$\frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} \cdots A_r^{\alpha_r}} = \frac{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_r)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2) \cdots \Gamma(\alpha_r)}$$
$$\times \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_r \frac{x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} \cdots x_r^{\alpha_r - 1} \delta(1 - x_1 - x_2 - \cdots - x_r)}{\left[x_1 A_1 + x_2 A_2 + \cdots + x_r A_r\right]^{\alpha_1 + \alpha_2 + \cdots + \alpha_r}}$$

Feynman parameter example

- Consider self-energy integral in scalar ϕ^3 theory
- Loop integral



$$\int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{1}{[k^2 - m^2][(k+p)^2 - m^2]} = \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^n k}{(2\pi)^n} \frac{1}{[k^2 + 2xp\cdot k + xp^2 - m^2]^2}$$

  - Can do the k-integral now, after completing the square $(k' = k + xp)$, using the standard formula

- Result

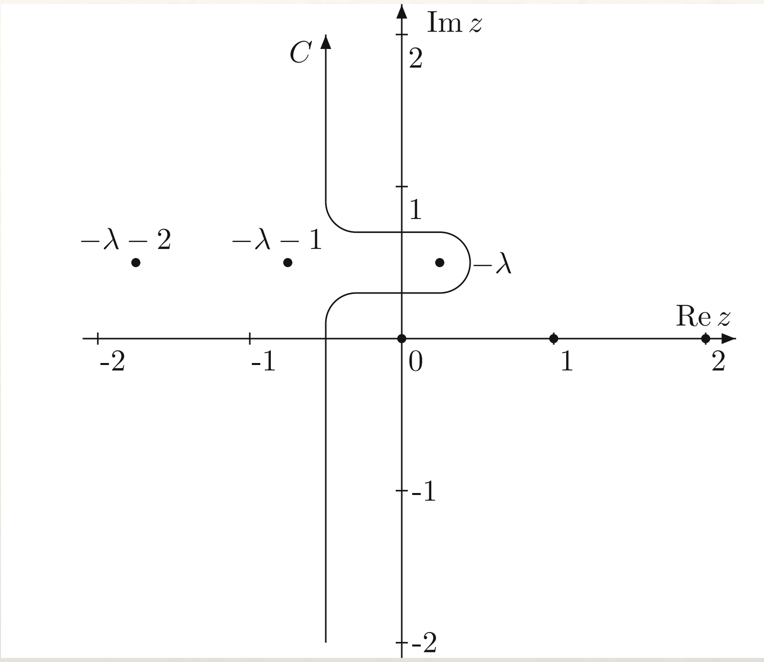
$$\frac{i\pi^{n/2}}{(2\pi)^n} \frac{\Gamma(2-n/2)}{\Gamma(2)} \int_0^1 \mathrm{d}x\,[m^2 - x(1-x)p^2]^{n/2-2}$$

- Dimensional regularisation $n = 4 - 2\epsilon$
  -

Mellin Barnes representation

- But for some integrals one would like to do the opposite, and factorize the expression

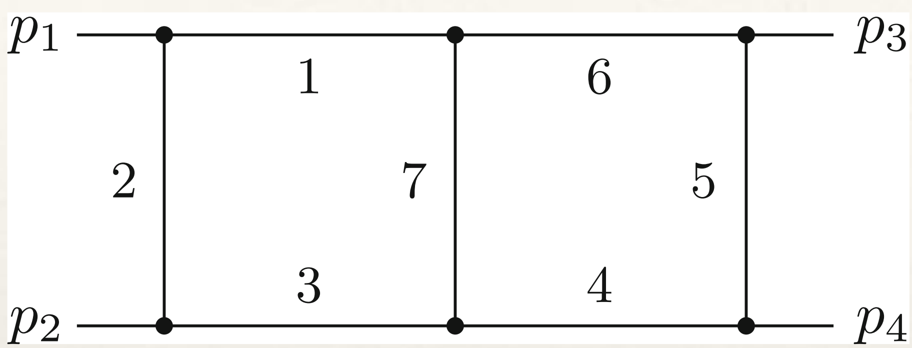
$$\frac{1}{(X+Y)^{\lambda}} = \frac{1}{2\pi \mathrm{i}\,\Gamma(\lambda)} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z\, \Gamma(\lambda+z)\Gamma(-z)\, \frac{1}{X^{\lambda+z}Y^{-z}}$$



  - With contour swinging between two series of poles
  - Contour can be closed either to right or left
- E.g. for $\lambda = 1$ write the left hand side as $\sum_{n=0}^{\infty} (-)^n \frac{Y^n}{X^{n+1}}$
- Close contour on right, so that result involves sum of residues of $\Gamma(-z)$ at $z = 0,1,2,..$
- Use that residue at $z = n$ equals $(-)^n/n!$, and identity follows
- MB representation can really help with computing difficult integrals

Double box

- A breakthrough case was the double box integral, computed by Smirnov in 1999

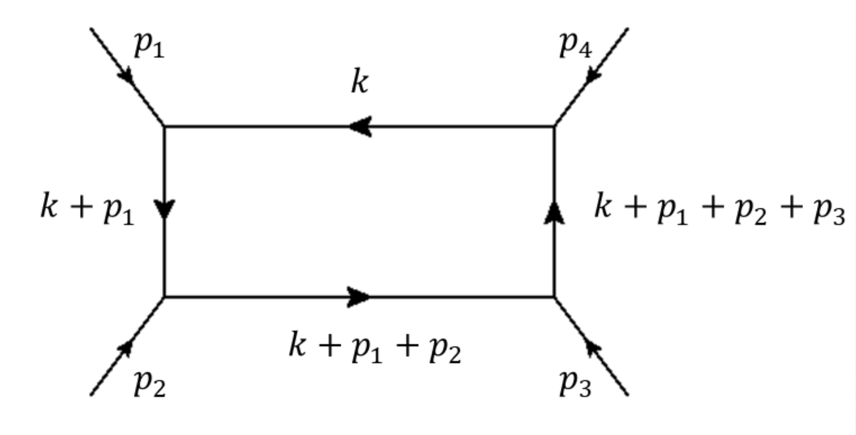


$$K(x,\epsilon) \quad = \quad K_{0t}(x,\epsilon) + K_{1t}(x,\epsilon) + o(\epsilon)$$

$$K_{0t}(x,\epsilon) \quad = \quad -\frac{4}{\epsilon^4} + \frac{5\ln x}{\epsilon^3} - \left(2\ln^2 x - \frac{5}{2}\pi^2\right)\frac{1}{\epsilon^2}$$
$$-\left(\frac{2}{3}\ln^3 x + \frac{11}{2}\pi^2 \ln x - \frac{65}{3}\zeta(3)\right)\frac{1}{\epsilon} + \frac{4}{3}\ln^4 x + 6\pi^2\ln^2 x - \frac{88}{3}\zeta(3)\ln x + \frac{29}{30}\pi^4$$

Differential equations for loops

- Method to compute loop diagrams as solutions to DE, with "simple" boundary conditions
- Example: 1-loop massless box integral



$$I_{box} = \frac{\mu^{2\epsilon}}{i\pi^{n/2} r_\Gamma} \int d^n k \frac{1}{k^2 (k+p_1)^2 (k+p_1+p_2)^2 (k+p_1+p_2+p_3)^2}$$

- Integral can be done using 3 Feynman parameters. Result

$$I_{box} = \frac{\mu^{2\epsilon}}{st} [\frac{2}{\epsilon^2}((-s)^{-\epsilon} + (-s)^{-\epsilon}) - \pi^2 - \ln^2(s/t) + O(\epsilon)]$$

$$s = 2p_1 \cdot p_2 \,, \quad t = 2p_1 \cdot p_3$$

- Define family of box integrals

$$G_{a_1 a_2 a_3 a_4} = \frac{\mu^{2\epsilon}}{i\pi^{n/2} r_\Gamma} \int d^n k \frac{1}{[k^2]^{a_1} [(k+p_1)^2]^{a_2} [(k+p_1+p_2)^2]^{a_3} [(k+p_1+p_2+p_3)^2]^{a_4}}$$

  - i.e. $I_{box} = G_{1111}$

IBP identities

- IBP = Integration By Parts
- Can derive recursion relations for $G_{a_1 a_2 a_3 a_4}$ using

$$0 = \int d^D k \, \frac{\partial}{\partial k^\mu} \, \xi^\mu \, \frac{1}{[k^2]^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}[(k+p_1+p_2+p_3)^2]^{a_4}}$$

- which leads to

$$(D - a1 - 2a2 - a3 - a4)\, G_{a1,a2,a3,a4} - t\, G_{a1,a2,a3,a4+1} - a1\, G_{a1+1,a2-1,a3,a4} - a3\, G_{a1,a2-1,a3+1,a4} - a4\, G_{a1,a2-1,a3,a4+1} = 0$$

  - and a few others

- There are then a few “Master integrals” from which the rest of the family can be derived

$$f_1 = G_{0101}\,, \quad f_2 = G_{1010}\,, \quad f_3 = G_{1111}$$

  - Notice that the first two are bubble integrals

Differential equation

- Define differential operator

$$\partial_s = (\beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3)\partial_{p_1} \qquad \beta_1 = \frac{2s+t}{2s(s+t)}\,, \quad \beta_2 = \frac{1}{2s}\,, \quad \beta_3 = \frac{1}{2(s+t)}$$

- Using IBP's one can derive

$$A_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\epsilon}{s} & 0 \\ \frac{-2(1-2\epsilon)}{st(s+t)} & \frac{2(1-2\epsilon)}{s^2(s+t)} & -\frac{s+t+\epsilon t}{s(s+t)} \end{pmatrix}$$

$$\partial_s \vec{f}(s,t,\epsilon) = A_s(s,t,\epsilon)\vec{f}(s,t,\epsilon)$$

- Choose a different (canonical, Henn) basis

$$g_1 = c(-s)^\epsilon t G_{0,1,0,2} = c(-s)^\epsilon (1-2\epsilon) G_{0,1,0,1}$$
$$g_2 = c(-s)^\epsilon s G_{1,0,2,0} = c(-s)^\epsilon (1-2\epsilon) G_{1,0,1,0}$$
$$g_3 = c\epsilon(-s)^\epsilon st G_{1,1,1,1}$$

- And use $x = t/s$ with $\partial_x = (-s^2/t)\partial_s$ to derive

$$\partial_x \vec{g}(x,\epsilon) = \epsilon\left[\frac{a}{x} - \frac{b}{1+x}\right]\vec{g}(x,\epsilon) \qquad a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix}$$

  - Note that the rhs is proportional to ϵ!
  - Such a basis can in very many cases be found

Solution to differential equation

- The canonical basis is the best one to solve the differential equation
- Bubble integrals:

$$G_{1,0,2,0} = \frac{\Gamma(1+\epsilon)}{\epsilon s(-s)^{\epsilon}} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

$$G_{0,1,0,2} = \frac{\Gamma(1+\epsilon)}{\epsilon t(-t)^{\epsilon}} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

  - i.e. g_1, g_2 known
- Now write the $g's$ as Laurent series in ϵ: $\vec{g} = \sum_{k\geq 0}^{\infty} \epsilon^k \vec{g}^{(k)}(x)$
  - and substitute into differential equation
- Because of the ϵ factor on the right, one has the structure

$$\partial_x g_3^{(i)} = F[g_k^{(i-1)}]$$

  - Special kinematic values for the master integrals are the boundary conditions.
  - Now integrate per i-value, to reconstruct the box integral g_3 to arbitrary power of ϵ as iterated integrals
- Highly powerful method, much used by the loop specialists

Iterated integrals

- Solutions contain iterated integrals
- The take the general form

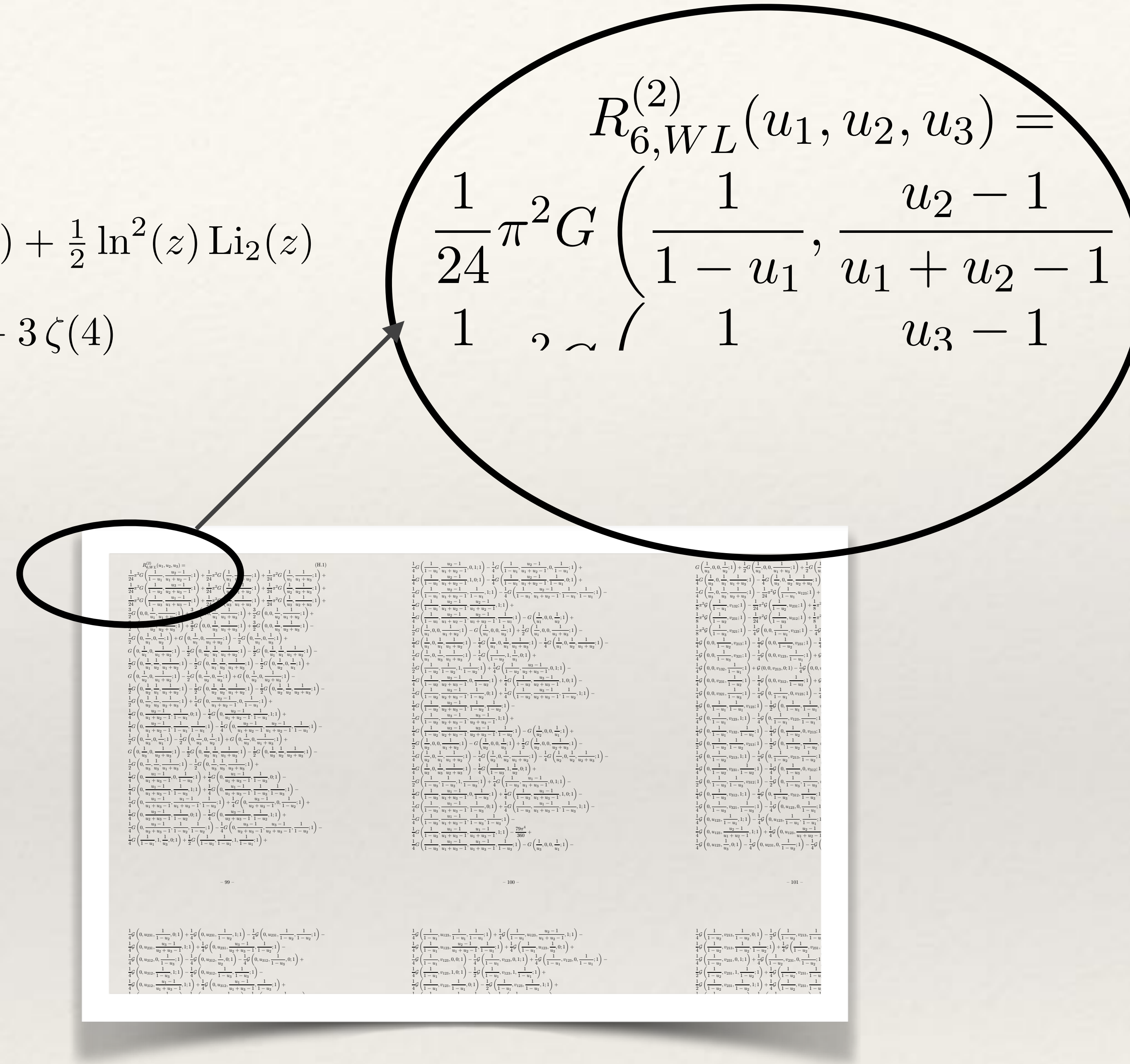
$$G(a_1, a_2, \ldots, a_n; z) = \int_0^z \frac{\mathrm{d}t_1}{t_1 - a_1} \int_0^{t_1} \frac{\mathrm{d}t_2}{t_2 - a_2} \ldots$$

  - "Goncharov" or multiple polylogarithms (MPL's)
- MPL's were recently found to obey interesting properties
  - shuffle algebra $(A \otimes A \to A)$
  - Hopf algebra with co-product $(A \to A \otimes A)$
- A bit formal, but very useful to find lots of identities among these functions

Special function relations can really help

- Example of such relations

$$G\left(\frac{1}{1-z},\frac{1}{1-z},1,\frac{1}{1-z};1\right) = 3\,\mathrm{Li}_4(z) - 2\ln(z)\,\mathrm{Li}_3(z) + \tfrac{1}{2}\ln^2(z)\,\mathrm{Li}_2(z) + \tfrac{1}{24}\ln^4(z) - \zeta(3)\ln(z) - 3\,\zeta(4)$$



- Remarkable benefit (2-loop hexagon function)
  - First result 17 pages of formulas, full of MPL's — Bern et al
  - After using MPL identities

$$R^{(2)}_{6,WL}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4\left(1-\frac{1}{u_i}\right)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2\left(1-\frac{1}{u_i}\right)\right)^2 + \frac{J^4}{24} + \chi\frac{\pi^2}{12}\left(J^2+\zeta(2)\right)$$

FREE SHIPPING

ON ALL ORDERS*

Predictive power in Quantum Field Theory

- Observable, computed in perturbation theory

$$\hat{O} = \sum_n c_n \, \alpha^n + R_n$$

- Finite order: only take lowest few "n". Then please complete the following checklist
  - ☑ α is small enough?
  - ☑ Is R_n small enough ?
  - ☑ c_n: does it not grow too fast with n?
- **Here we worry about the last check.**
- Hadronic observable is then convolution of PDFs and partonic cross section

$$O_H = \sum_{i=q,..} \phi_i(\mu) \otimes \hat{O}_i(\mu)$$

  - ☑ If any of these steps is a weak link, try to update it

Perturbative series in QFT

- Typical perturbative behavior of observable
  - α is the coupling of the theory (QCD, QED, ..)
  - L is some numerically large logarithm
  - "1" = π^2, ln(2), anything not-logarithmic
  - Notice: ***effective* expansion parameter is αL^2 i.e. a problem when >1!!**
  - **Fix**: reorganize/resum terms such that

$$\hat{O}_2 = 1 + \alpha(L^2 + L + 1) + \alpha^2(L^4 + L^3 + L^2 + L + 1) + \ldots$$

$$\begin{aligned}\hat{O} &= 1 + \alpha_s(L^2 + L + 1) + \alpha_s^2(L^4 + L^3 + L^2 + L + 1) + \ldots \\ &= \exp\left(\underbrace{\underbrace{Lg_1(\alpha_s L)}_{LL} + g_2(\alpha_s L)}_{NLL} + \alpha_s g_3(\alpha_s L) + \ldots \right) \underbrace{C(\alpha_s)}_{\text{constants}} \\ &+ \text{ suppressed terms}\end{aligned}$$

- Notice the definition of LL, NLL, etc

LL, NLL,.. and matching to fixed order

- Leading-log, next-to-leading log, etc
  - Schematic overview

$$O = \alpha_s^p \left(\underbrace{\underbrace{C_0}_{\text{LL,NLL}} + C_1 \alpha_s}_{\text{NNLL}} + \dots \right) \exp\left[\underbrace{\underbrace{\underbrace{(\sum_{n=1} \alpha_s^n L^{n+1} c_n)}_{\text{LL}} + (\sum_{n=1} \alpha_s^n L^n d_n)}_{\text{NLL}} + (\sum_{n=1} \alpha_s^n L^{n-1} e_n)}_{\text{NNLL}} + \dots \right]$$

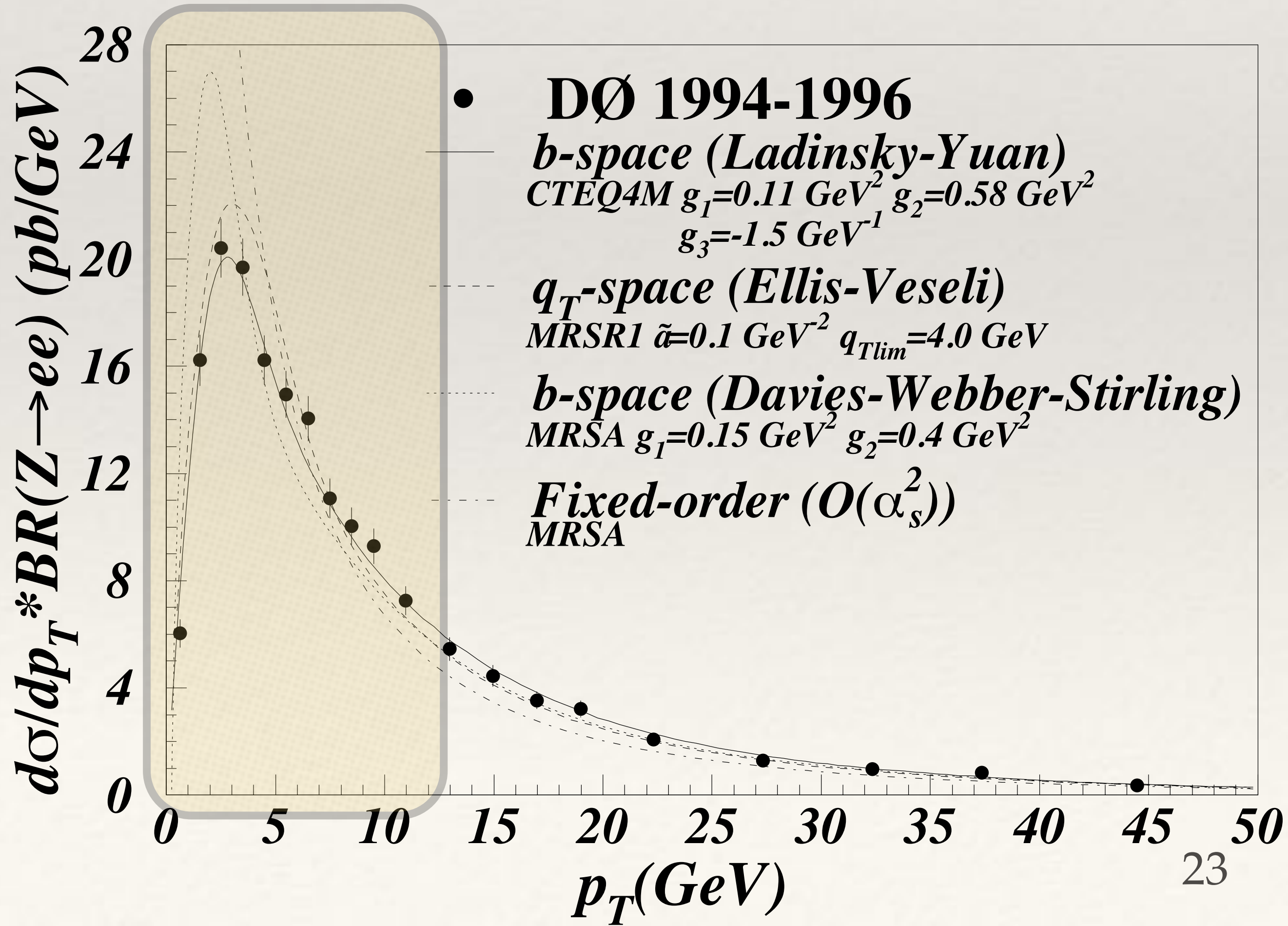
  - Systematic expansion in α_s in the exponent
    - ✓ If we can find the coefficients c_n, d_n, e_n, C_0, C_1 etc
  - How to combine this with an exact NLO or NNLO calculation?

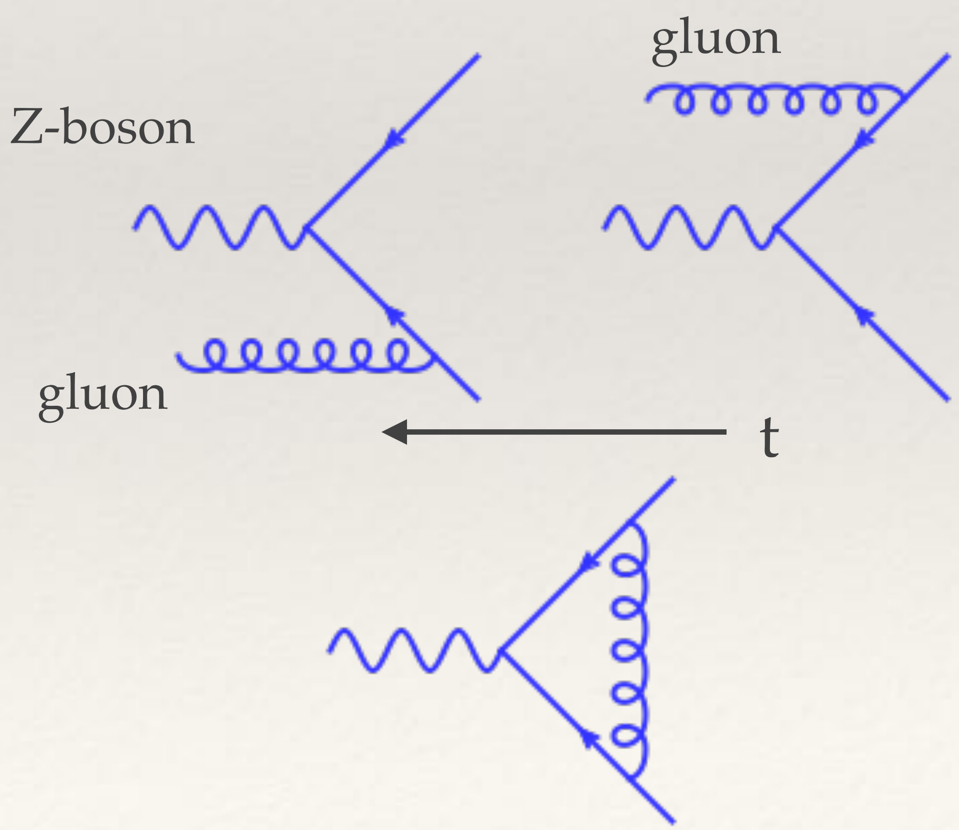
$$O_{\text{NLO matched}} = O_{\text{NLO}} + O_{\text{resummed}} - (O_{\text{resummed}})\big|_{\text{expanded to } \mathcal{O}(\alpha_s)}$$

Logs of what? First case: double recoil logs

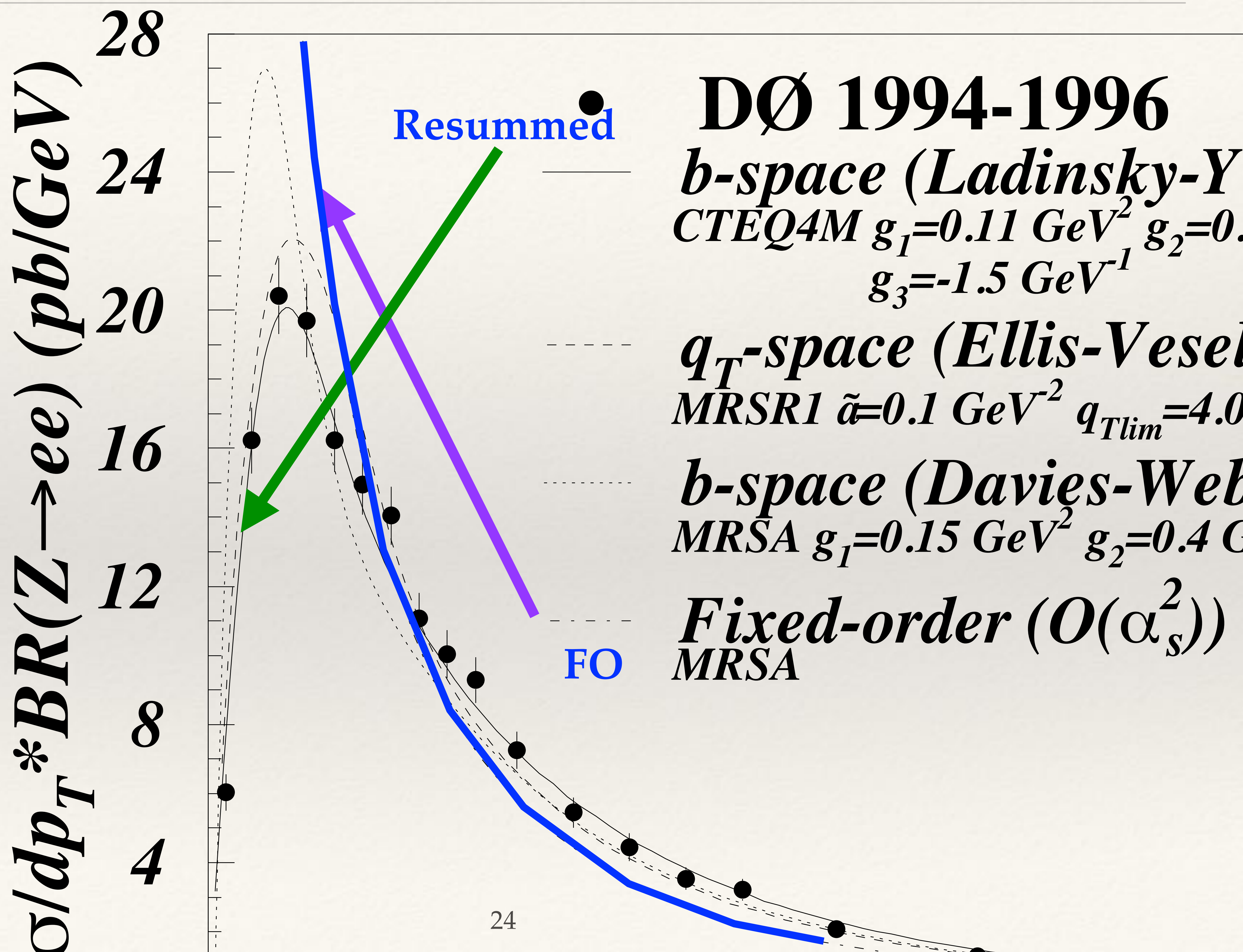
$$L^2 = \ln^2\left(p_T^2/M_Z^2\right)$$

- Eg. pT of Z-bosons produced in hadron collisions
  - Z-boson gets p_T from recoil agains (soft) gluons
    - ✓ 1 emission with gluon very soft: divergent
  - Typical: turn-over at pT around 5 GeV is **only explained by resummation**, not by any finite order calculation





Recoil logs

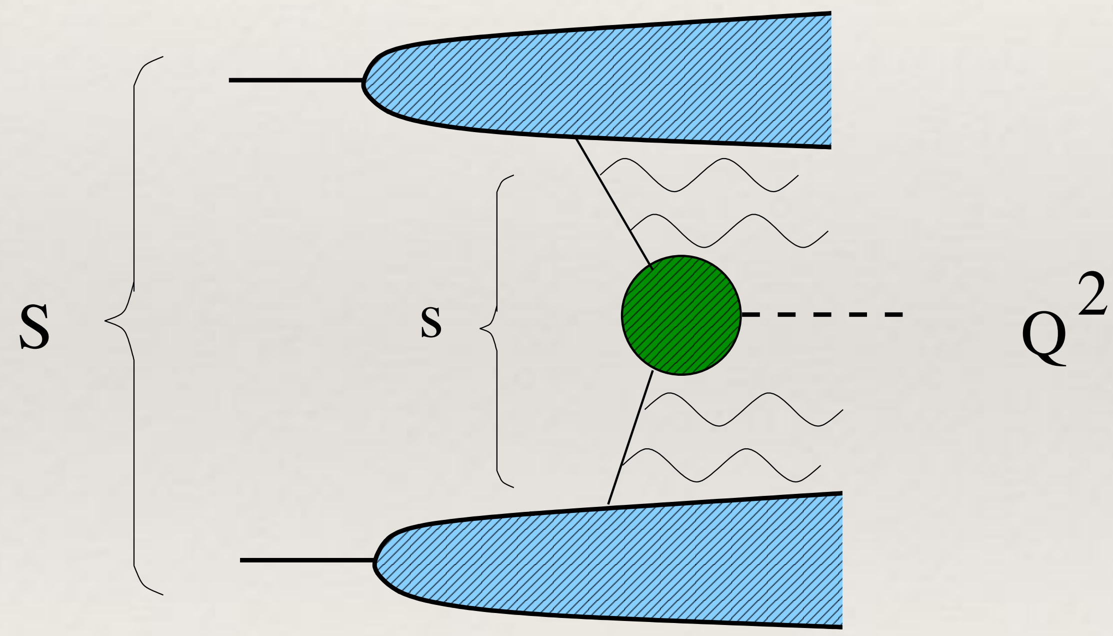


DØ 1994-1996
Resummed
b-space (Ladinsky-Y
CTEQ4M g_1=0.11 GeV^2 g_2=0.
g_3=-1.5 GeV^{-1}
q_T-space (Ellis-Vesel
MRSR1 $\tilde{a}$=0.1 GeV^{-2} q_{Tlim}=4.0
b-space (Davies-Web
MRSA g_1=0.15 GeV^2 g_2=0.4 G
Fixed-order ($O(\alpha_s^2)$)
MRSA
FO
$d\sigma/dp_T$*BR($Z\rightarrow ee$) (pb/GeV)
28
24
20
16
12
8
4

Second case: threshold logarithms

$$L^2 = \ln\left(1 - \frac{Q^2}{s}\right) \equiv \ln^2(1-z)$$

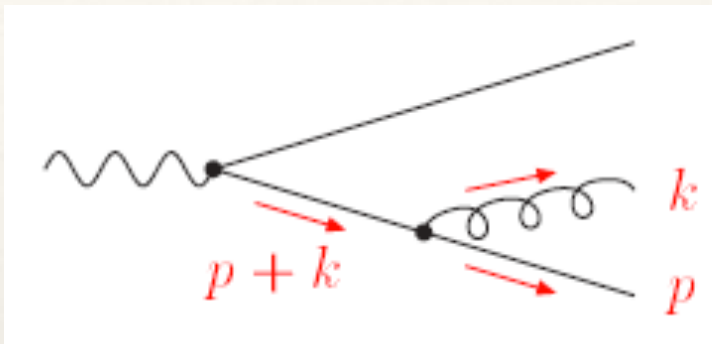
- Logarithm^2 of "energy above threshold Q^2"
  - "Hidden" logs": have *integration variables* in arguments
  - Typical effect: enhancement of cross section



$$S \geq s \geq Q^2$$

Quick look at origin of double ("Sudakov") logs

- Double logarithms in cross sections are related to IR divergences

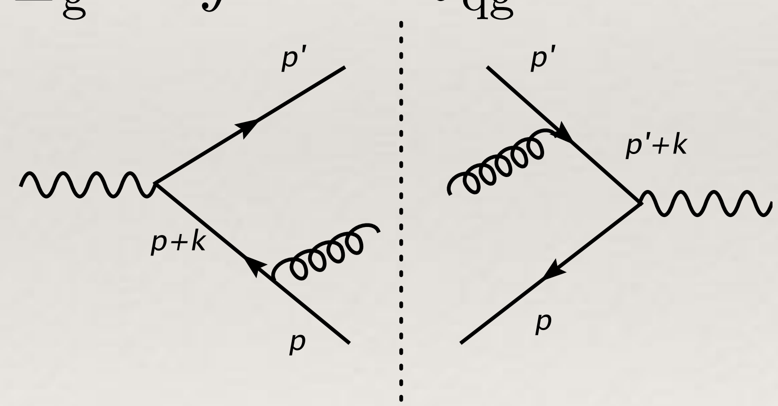


$$\frac{1}{(p+k)^2} = \frac{1}{2p\cdot k} = \frac{1}{2E_{\rm g}E_{\rm q}(1-\cos\theta_{\rm qg})}$$

Phase space integration

$$\alpha_s \int \frac{d^{4-2\epsilon}k}{(2\pi)^4}\,\frac{p\cdot p'}{p\cdot k\; p'\cdot k} \sim \alpha_s \int^K \frac{dE_{\rm g}E_{\rm g}^{-\epsilon}}{E_{\rm g}} \int \frac{d\theta_{\rm qg}\sin^{-\epsilon}\theta_{\rm qg}}{\theta_{\rm qg}}$$

$$\sim \alpha_s\,(\frac{1}{\epsilon^2} + \ln^2(K))\,.$$



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Benefits of resummation

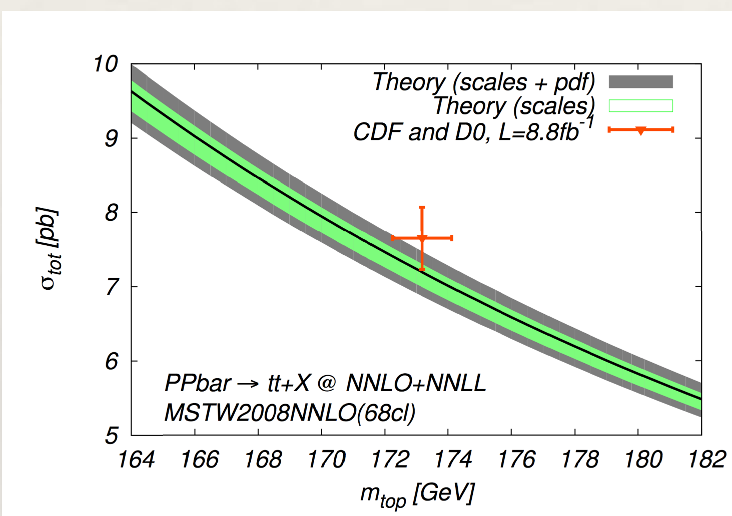
- It can rescue predictive power
  - when perturbative series converges poorly
- Better physics description (small p_T e.g., more later)
- Lessens the renormalization/factorization scale uncertainty,
  - e.g. the inclusive top quark and Higgs cross section:

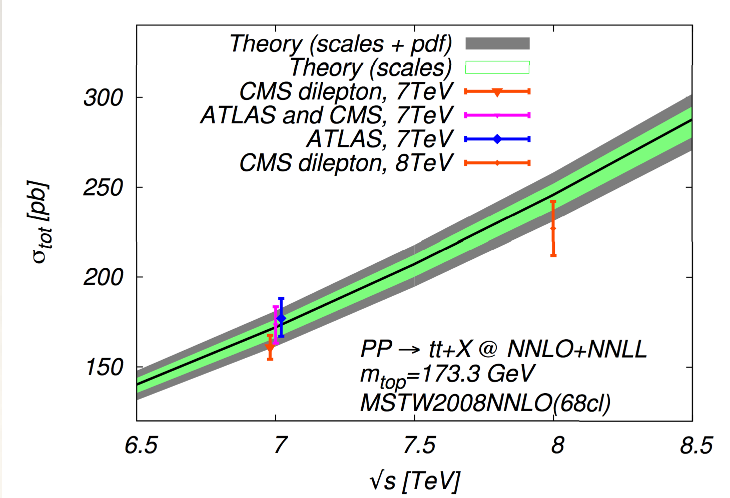
NNLO-NNLL inclusive tt cross section

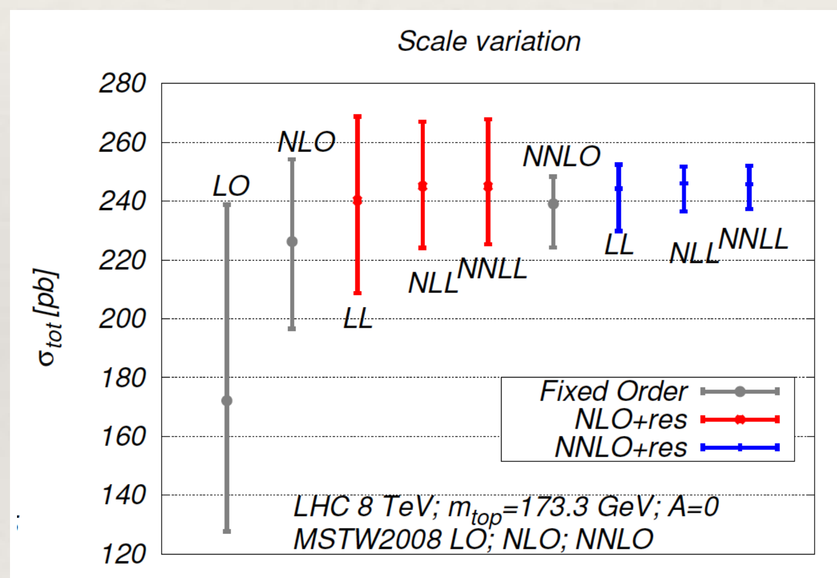
Baernreuther, Fiedler, Mitov, Czakon

- Highly impressive effort
  - For precision top physics
  - useful for gluon density at large x, and α_S determination

Czakon, Mitov, Mangano, Rojo







Concurrent uncertainties:

Scales	~ 3%
pdf (at 68%cl)	~ 2-3%
α_S (parametric)	~ 1.5%
m_{top} (parametric)	~ 3%

Soft gluon resummation makes a difference

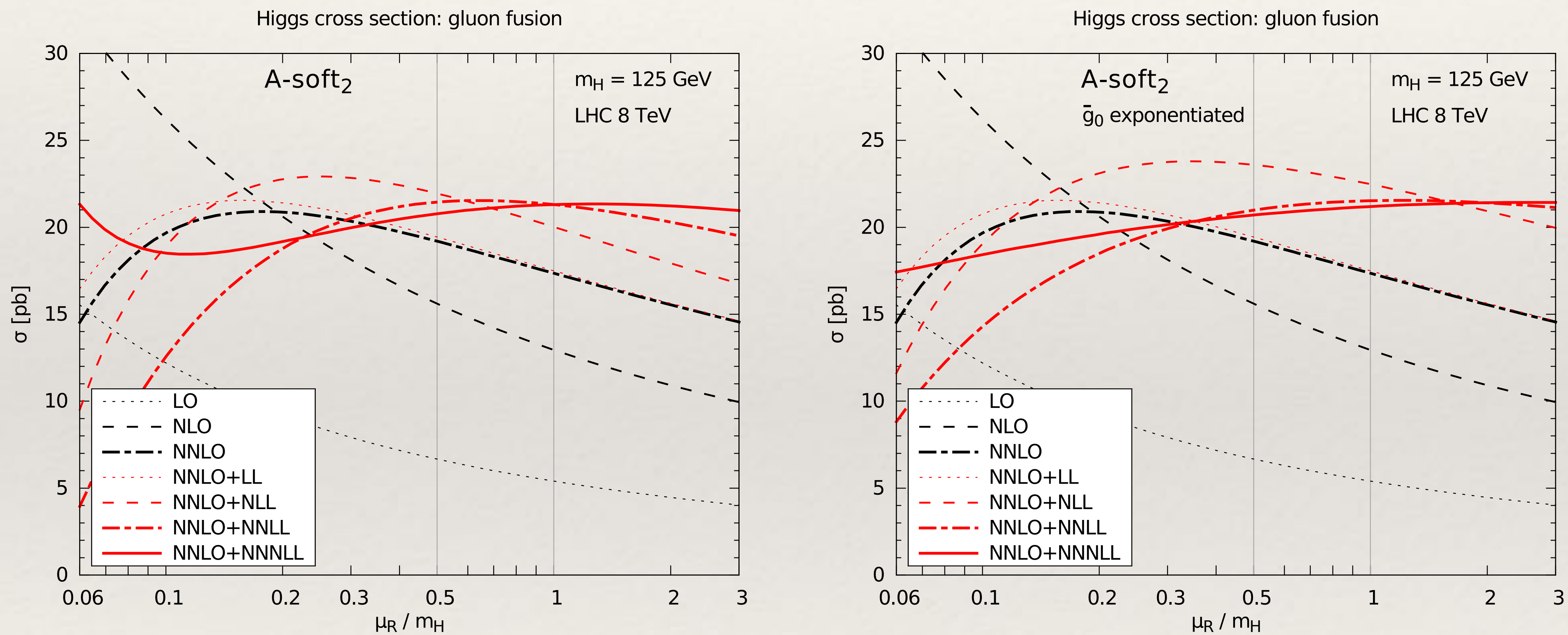
5% -> 3%

N^3LL resummation for Higgs production

- Logarithm is again threshold logarithm

Bonvini, Marzani

  - For inverse Mellin transform, employ both Minimal Prescription and Borel prescription
  - Nice progression, especially with exponentiated constants



  - Code: ResHiggs and ggHigs

How to resum?

- There are many ways, depending on
  - the observable
  - the logarithm
  - the resummer
- Here we take as key notions
  - approximations in kinematic limits
  - factorization

Resummation 101

- Cross section for n extra gluons

Phase space measure **Squared matrix element**

$$\sigma(n) = \frac{1}{2s}\int d\Phi_{n+1}(P,k_1,\ldots,k_n) \times |\mathcal{M}(P,k_1,\ldots,k_n)|^2$$

- When emissions are soft, can factorize phase space measure and matrix element [eikonal approximation]

$$d\Phi_{n+1}(P,k_1,\ldots,k_n) \longrightarrow d\Phi(P) \times \Big(d\Phi_1(k)\Big)^n \frac{1}{n!}$$

- Sum over all orders

$$|\mathcal{M}(P,k_1,\ldots,k_n)|^2 \longrightarrow |\mathcal{M}(P)|^2 \times \Big(|\mathcal{M}_{1\ \mathrm{emission}}(k)|^2\Big)^n$$

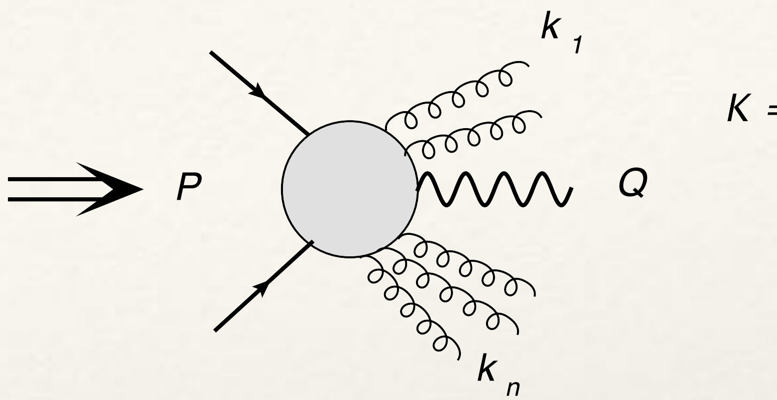
$$\sum_n \sigma(n) = \sigma(0) \times \exp\left[\int d\Phi_1(k)|\mathcal{M}_{1\ \mathrm{emission}}(k)|^2\right]$$

- Incorporate Theta or Delta functions in space space
  - but these must factorize similarly, or they cannot go into exponent

Phase space in resummation

- Kinematic condition expresses "z" in terms of gluon energies

$$s = Q^2 - 2P\cdot K - K^2 \qquad \delta\Big(1 - \frac{Q^2}{s} - \sum_i \frac{2k_i^0}{\sqrt{s}}\Big)$$



  - or conservation of transverse momentum

$$\delta^2(Q_T - \sum_i p_T^i)$$

- Transform (e.g. Laplace/Mellin or Fourier) factorizes the phase space

$$\int_0^\infty dw\, e^{-w\,N}\delta\Big(w - \sum_i w_i\Big) = \prod_i \exp(-w_i N) \qquad \int d^2Q_T\, e^{ib\cdot Q_T}\, \delta^2(Q_T - \sum_i p_T^i) = \prod_i e^{ib\cdot p_T^i}$$

- So can go into exponent

$$\sum_n \sigma(n) = \sigma(0) \times \exp\left[\int d\Phi_1(k)|\mathcal{M}_{1\ \mathrm{emission}}(k)|^2(\exp(-w\,N) - 1)\right]$$

  - Large logs: ln(N) or ln(bQ)

Resummation from factorization

- Very generically, if a quantity factorizes, one can resum it
  - First a toy example: product of two function, with x and μ in common:

$$C(a,b,x) = f(a,x,\mu) \times g(b,x,\mu)$$

  - Take logarithm

$$\ln C(a,b,x) = \ln f(a,x,\mu) + \ln g(b,x,\mu)$$

  - Take derivative w.r.t. μ

$$\mu \frac{d}{d\mu} \ln f(a,x,\mu) = -\mu \frac{d}{d\mu} \ln g(b,x,\mu) \equiv \gamma(x,\mu)$$

    - Note: γ can only depend on common variables
  - Solve

$$f(a,x,\mu) = f(a,x,\mu_0) \exp\left[-\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma(x,\mu')\right]$$

    - Resummation!

Resummation and factorization

- Factorization is actually separation of degrees of freedom
  - Renormalization; factorizes UV modes into Z-factor

$$G_B(g_B,\Lambda,p) = Z\Big(\frac{\Lambda}{\mu}, g_R(\mu)\Big) \times G_R\Big(g_R(\mu), \frac{p}{\mu}\Big)$$

  - Evolution equation (here RG equation)

$$\mu\frac{d}{d\mu}\ln G_R\Big(g_R(\mu), \frac{p}{\mu}\Big) = -\mu\frac{d}{d\mu}\ln Z\Big(\frac{\Lambda}{\mu}, g_R(\mu)\Big) = \gamma(g_R(\mu))$$

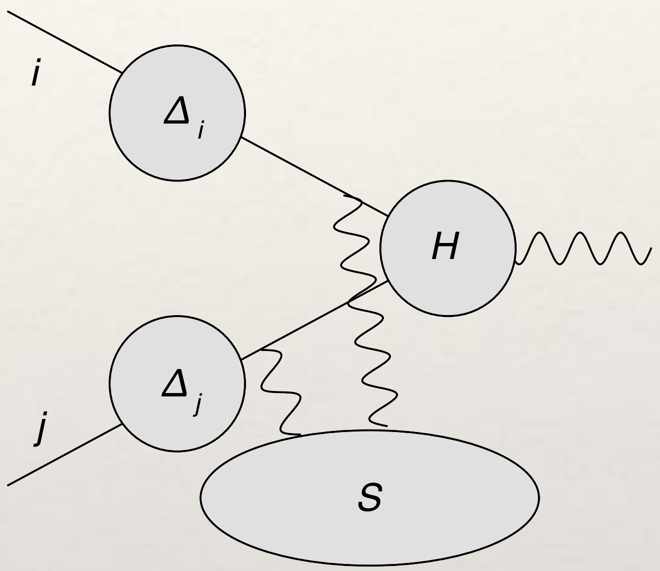
  - Solving = resumming

$$G_R\left(\frac{p}{\mu}, g_R(\mu)\right) = G_R\left(1, g_R(p)\right) \underbrace{\exp\left[\int_p^\mu \frac{d\lambda}{\lambda}\gamma(g_R(\lambda))\right]}_{\text{resummed}}$$

Factorization and resummation for Drell-Yan

$$\sigma(N) = \Delta(N,\mu,\xi_1)\Delta(N,\mu,\xi_2)S(N,\mu,\xi_1,\xi_2)H(\mu)$$

- Near threshold, cross section is equivalent to product of 4 well-defined functions
- Demand independence of
  - renormalization scale μ
  - gauge dependence parameter ξ
    - ✓ find exponent of double logarithm



$$0 = \mu\frac{d}{d\mu}\sigma(N) = \xi_1\frac{d}{d\xi_1}\sigma(N) = \xi_2\frac{d}{d\xi_2}\sigma(N)$$

Contopanagos, EL, Sterman

$$\Delta = \exp[\int\frac{d\mu}{\mu}\int\frac{d\xi}{\xi}..]$$

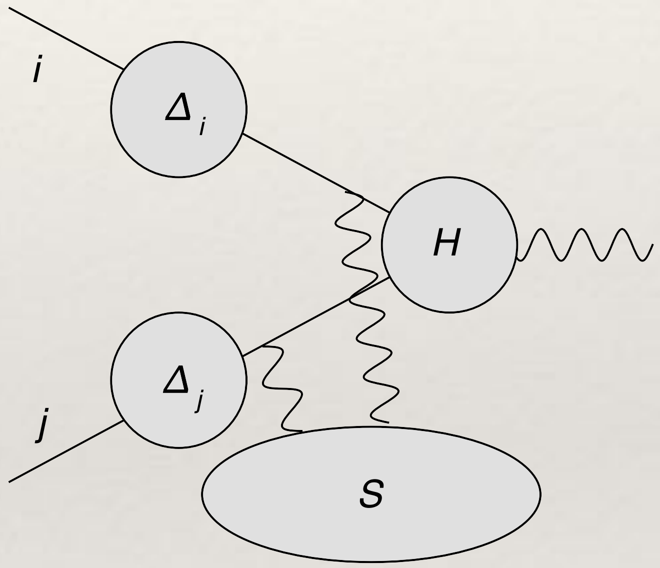
Factorization for threshold resummation

- $\Delta_i(N)$: initial state soft+collinear radiation effects
  - real+virtual
  - $\alpha_s^n \ln^{2n} N$

$$\sigma(N) = \sum_{ij} \phi_i(N)\phi_j(N) \times \underbrace{\left[\Delta_i(N)\Delta_j(N)S_{ij}(N)\,H_{ij}\right]}_{\hat{\sigma}_{ij}(N)}$$

- $S_{ij}(N)$: soft, non-collinear radiation effects
  - $\alpha_s^n \ln^n N$

- H: hard function, no soft and collinear effects



$$\Delta_i(N) = \exp\left[\ln N \frac{C_F}{2\pi b_0 \lambda}\{2\lambda + (1-2\lambda)\ln(1-2\lambda)\} + ..\right]$$
$$= \exp\left[\frac{2\alpha_s C_F}{\pi}\ln^2 N + ..\right]$$

Threshold resummed Drell-Yan/Higgs cross section

Sterman; Catani, Trentadue

$$
\begin{aligned}
\frac{d\sigma^{\text{resum}}}{dQ^2}(z) &= \int_C \frac{dN}{2\pi\, i}\, z^{-N}\, \hat{\sigma}(N) \\
\sigma(N) &= \exp\left[-\int_0^1 dx\, \frac{x^{N-1}-1}{1-x} \left\{ \int_{Q^2}^{Q^2(1-x)^2} \frac{d\mu}{\mu} A(\alpha_s(\mu)) \right.\right. \\
&\qquad \left.\left. + D(\alpha_s((1-x)Q)) \right\} \right] \times \left(1 + \alpha_s(Q^2)\frac{C_F}{\pi} + \ldots\right)
\end{aligned}
$$

Note: functions in exponent only depend on α_s

Resummation and factorization

- Type of factorization dictates resummation
  - small x [ln(x)] $\longrightarrow$ k_T factorization
    - ✓ Regge, High-Energy,..
  - large x [$\ln^2(1-x)$] $\longrightarrow$ near-threshold factorization
    - ✓ Threshold, Sudakov
- Systematic approach in Soft Collinear Effective Theory [SCET]
  - powerful framework, with many results

Bauer, Fleming, Pirjol, Stewart, ..

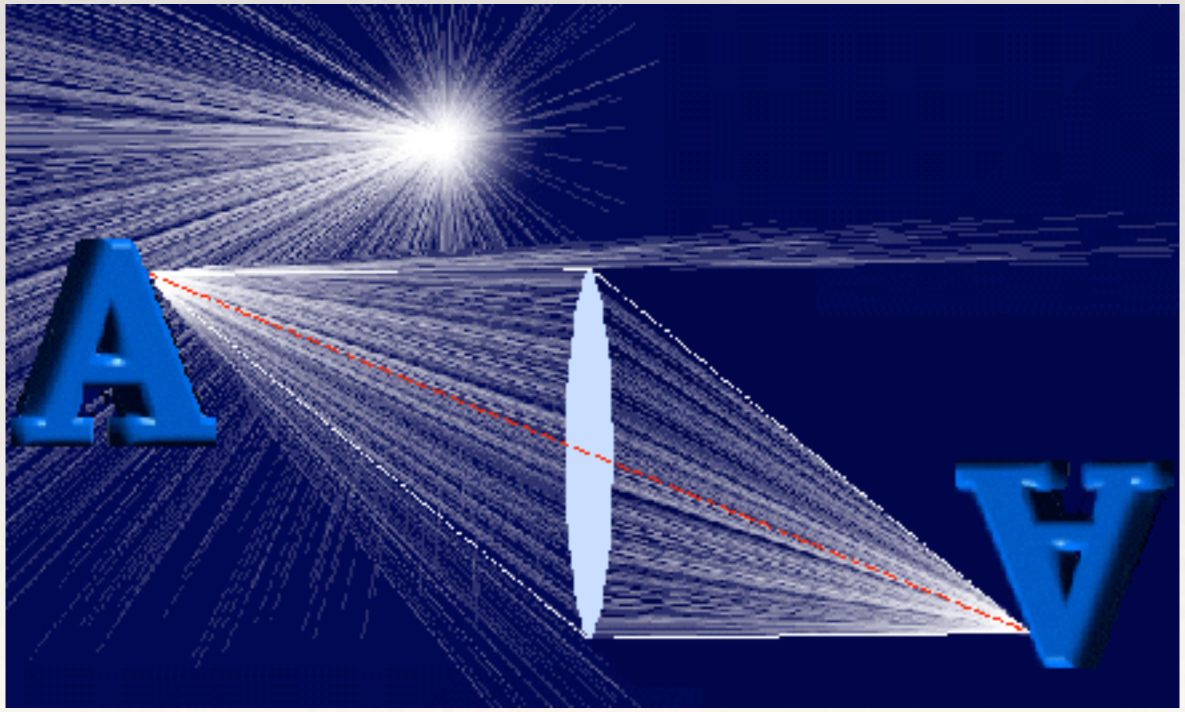
Beneke, Chapovsky, Diehl, Feldmann

Becher

Background: the eikonal approximation

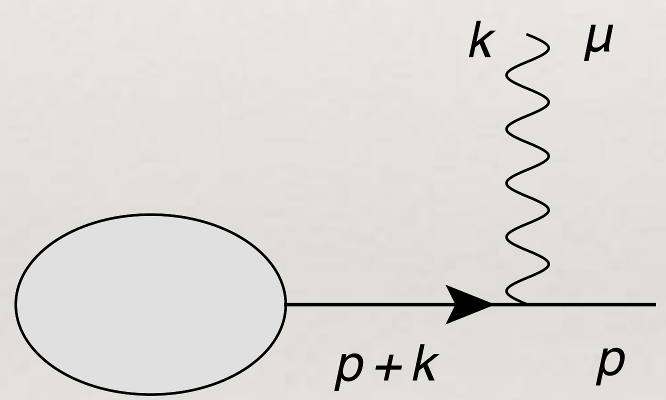
Eikonal optics: rays

- Can describe formation of images/eikons
  - wavelength << size of scatterer
- Cannot describe diffraction, polarization etc
  - these are wave phenomena
- In quantum field theory the eikonal approximation reveals more



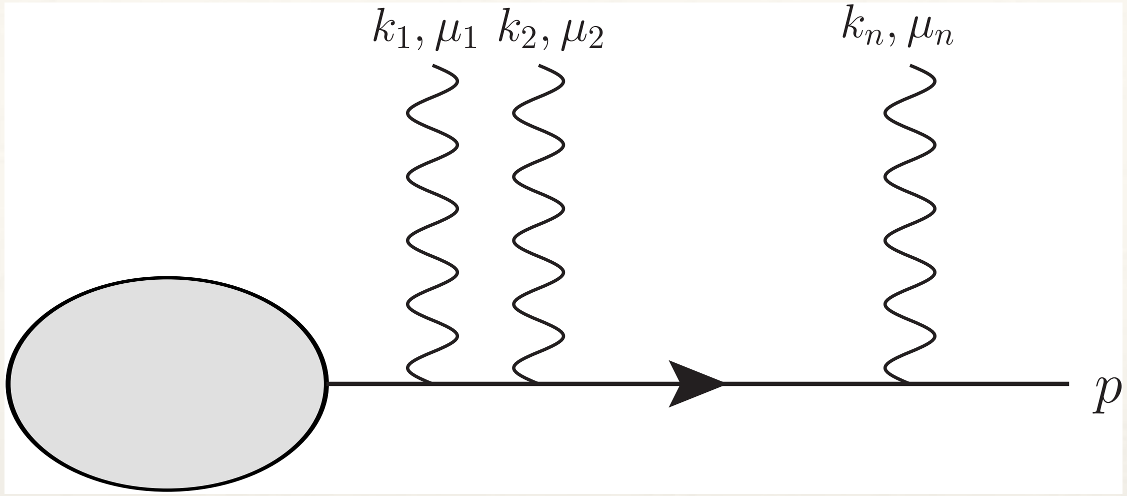
Eikonal approximation in QED

- Charged particle emits soft photon
  - Propagator: expand numerator & denominator in soft momentum, keep lowest order
  - Vertex: expand in soft momentum, keep lowest order



$$\frac{(p+k)^\mu + p^\mu}{2p\cdot k + k^2} \longrightarrow \frac{2p^\mu}{2p\cdot k}$$

Basics of eikonal approximation in QED



Exact: $$\frac{1}{(p+K_1)^2}(2p+K_2+K_1)^{\mu_1}\dots\frac{1}{(p+K_n)^2}(2p+K_n)^{\mu_n}, \quad K_i=\sum_{m=i}^{n} k_m.$$

Approx: $$\frac{1}{2pK_1}2p^{\mu_1}\dots\frac{1}{2pK_n}2p^{\mu_n}$$

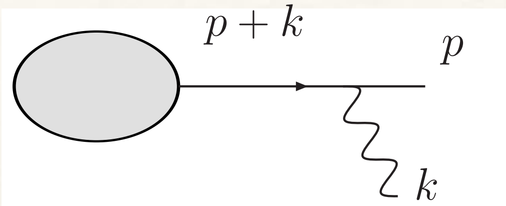
Eikonal identity: $$\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_2}+\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_1}=\frac{1}{p\cdot k_1\,p\cdot k_2}$$

Sum over all perm's: $$\prod_i \frac{p^{\mu_i}}{p\cdot k_i}.$$

Independent, uncorrelated emissions, Poisson process

Eikonal approximation: no dependence on emitter spin

- Emitter spin becomes irrelevant in eikonal approximation
  - Fermion



$$M\Big(\frac{i(\not p+\not k)}{(p+k)^2}\Big(-ig_s\gamma^\mu\Big)\,u(p)$$

  - Approximate, and use Dirac equation $\not p\, u(p) = 0$
  - Result:

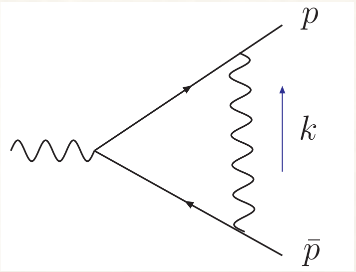
$$g\,(M\,u(p)) \times \frac{p^\mu}{p\cdot k}$$

  - Two things have happened
    - ✓ No sign of emitter spin anymore
    - ✓ Coupling of photon proportional to p^μ !
- Decoupling again of emission and emitter

Eikonal exponentiation

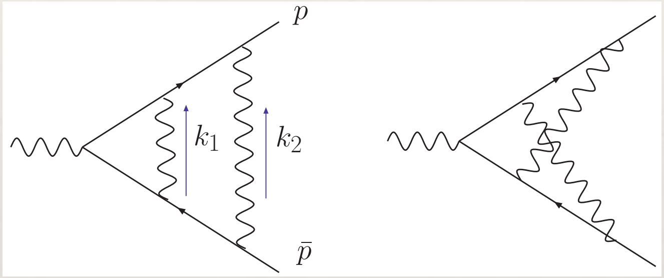
- In the eikonal approximation, suddenly we see very interesting patterns.

One loop vertex correction, in eikonal approximation



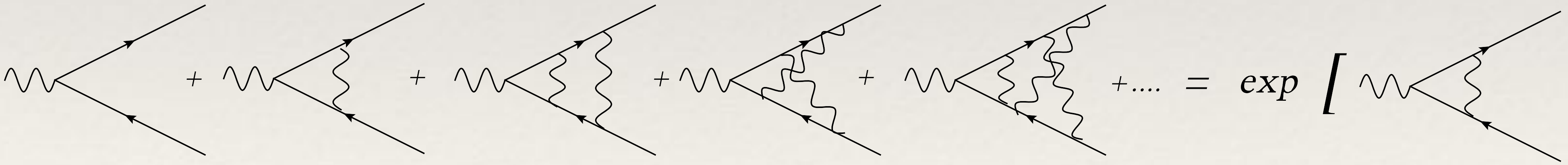
$$\mathcal{A}_0 \int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)}$$

Two loop vertex correction, in eikonal approximation



$$\mathcal{A}_0 \, \frac{1}{2} \left(\int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)} \right)^2$$

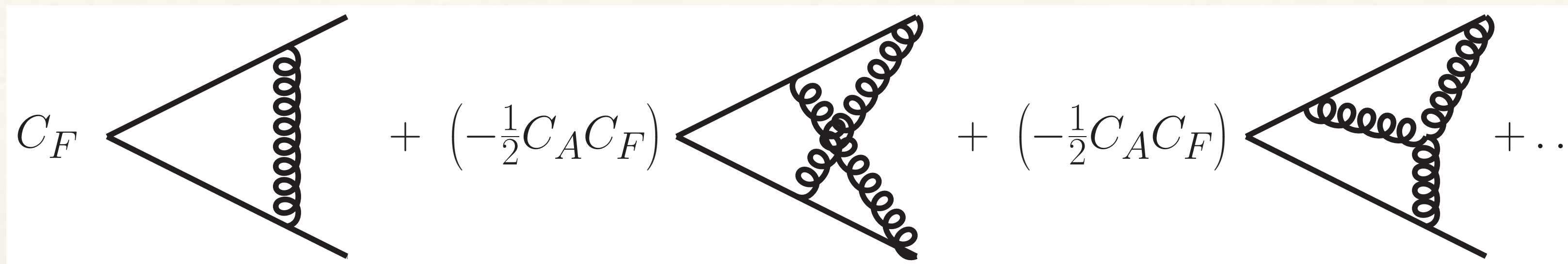
Exponential series! A really beautiful result



Yennie, Frautschi, Suura

QCD exponentiation: webs

Exp [



]

- Not immediately generalizable to QCD, seemingly
  - Vertices terms have color charges, which don't commute
  - Still, an exponentiation theorem holds

Webs

$$\sum_D \mathcal{F}_D C_D = \exp\left[\sum_i \bar{C}_i w_i\right]$$

Gatheral; Frenkel, Taylor; Sterman

EL, Stavenga, White

- Generalized to multiple colored external lines

  Gardi, EL, Stavenga, White

  Mitov, Sterman, Sung
  - For N(N..)LL resummation for jet cross sections, e.g.

Soft logarithms at next-to-leading power

- General soft expansion for 2→1 processes

$$\frac{d\sigma}{dz} = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{2n-1} \left\{ c_{nm}^{(-1)} \frac{\log^m(1-z)}{1-z}\bigg|_+ + c_{nm}^{(0)} \log^m(1-z) + \dots \right\}$$

- First term resummed well understood (NNNLL etc)
- NLP logarithms $\log^i(1-x)$
  - also exhibit all-order patterns. Leading logarithmic (only..) resummation now achieved for a number of reacions

$$\hat{\sigma}_{\rm LO}^{(gg)}(Q^2) \exp\left\{\frac{2\alpha_s C_A}{\pi}\log^2(N)\right\}\left(1 + \frac{2\alpha_s C_F}{\pi}\frac{\log N}{N}\right)$$

Beneke, Broggio, Garny, Jaskiewicz, Vernazza, Szafron, Wang '18
Bahjat-Abbas, Bonocore, EL, Magnea, Sinninghe-Damsté, Vernazza, White '19
Moult, Stewart, Vita, Xhu '18

QCD precision quest

- Development and applications of many new methods for high order partonic cross sections
  - Differential equations, Mellin-Barnes, Loop-tree duality, finite field methods, unitarity methods
  - (Isolate IR divergences in radiative contributions)
  - Many specialized computer codes, numerical or algebraic (FORM e.g.)
  - Resum to all orders what you can, and match to exact finite orders
- Improvement of PDFs and their uncertainty
  - New methods, better tools, better input data.
  - Delicate interplay of measurements and theory
- Improve the parton showers
  - Beyond leading logarithmic
  - Match to finite order calculations
- [Improve the hadronization models, jet algorithms etc]

Summary

- It is possible to use perturbation theory for QCD@LHC, thanks to asymptotic freedom
  - So can use LHC as precision machine, to seek small deviations from Standard Model
- Make every link in the chain (PDF’s, partonic cross section, parton showers) as precise as possible
- Much, much effort and ingenuity in computing higher order corrections, including automization
- Confront precise pQCD description with many different measurements to stress-test the Standard Model
  - Include also electroweak corrections..
- Theorists and experimenters both on Team Precision!