Topological strings and 2d conformal field theory

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How can we understand non-perturbative QFT?

- **Experiment.** Requires an explanation.
- **Lattice calculations.** Huge computational resources are needed. Still not an example of "understanding".
- **Study toy models.** Not directly relevant for experiments, but usually exactly solvable. Lots of fun!

Surprising (or not) observation

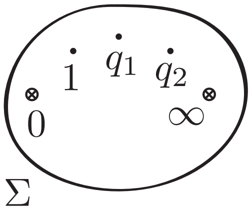
Many exactly solvable systems are related by nontrivial dualities.

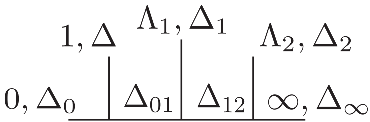
Perhaps this only means that we are always solving one and the same system in many different ways...

Crash course in $2d$ conformal field theory

Describes **phase transitions** in condensed matter, worldsheet theory of strings, etc. [Belavin, Polyakov, Zamolodchikov, 1984]

- Very large symmetry group — infinite dimensional Virasoro algebra, $[L_m, L_n] = (n-m)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$
- Primary fields V_Δ: $L_n V_\Delta = 0$, $n > 0$, $L_0 V_\Delta = \Delta V_\Delta$.
- Correlators factorise into products of holomorphic blocks: $\langle V(0,0)V(1,1)V(\Lambda_1,\bar{\Lambda}_1)V(\Lambda_2,\bar{\Lambda}_2)V(\infty,\infty)\rangle =$ $\sum_\Delta C_{\Delta,\ldots}\mathcal{B}(\Delta|\Lambda_1,\Lambda_2)\bar{\mathcal{B}}(\bar{\Delta}|\bar{\Lambda}_1,\bar{\Lambda}_2)$. $\mathcal{B}$ described by a comb diagram





$\mathcal{B}(\Delta_i|\Lambda_1,\Lambda_2)$

q-deformation of conformal field theory

The basic idea of q-deformation:

$$x = \underbrace{1+1+\ldots+1}_{x \text{ times}} \to 1 + q + \ldots + q^{x-1} = \frac{1-q^x}{1-q}$$

Gives rise to quantum groups and integrable systems.

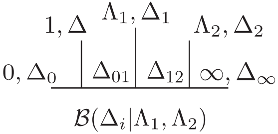
- Deformation of the Virasoro algebra:

$$[L_n, L_m] = -\sum_{l \geq 1} f_l (L_{n-l} L_{m+l} - L_{m-l} L_{n+l}) - \\ - \frac{(1-q)(1-t^{-1})}{(1-q/t)} \left((q/t)^n - (t/q)^n \right) \delta_{n+m,0}$$

- Primary fields V_Δ: $L_n V_\Delta = 0$, $n > 0$, $L_0 V_\Delta = \Delta V_\Delta$.
- Correlators still factorise but conformal blocks are deformed: $\widetilde{\mathcal{B}}(\Delta | \Lambda_1, \Lambda_2)$.

AGT correspondence and $\mathcal{N}=2$ gauge theories

$\mathcal{B}(\Delta_i|q_1,q_2) =$



$=$ Partition function of 4d $\mathcal{N}=2$ gauge theory with gauge group $SU(2)^2$ + some matter

The dictionary

- q_1 and q_2 are complexified coupling constants of two $SU(2)$ groups
- Δ_i are masses and Coulomb moduli of the gauge theory
- Central charge $c = 1 + 6(b+1/b)^2$ is related to ϵ-deformation parameters of the gauge theory $\epsilon_1 = 1, \epsilon_2 = b^2$.

q-deformation and $5d$ gauge theory

If the CFT is q-deformed, then $4d$ gauge theory is turned into $5d$ gauge theory on a circle of radius $R_5 = \log q$.

Topological strings

Ordinary strings

- Are of finite length $\Rightarrow$ no local field theory description (only at low energies)
- Can be described by an *infinite* number of interacting particle species with arbitrary high masses

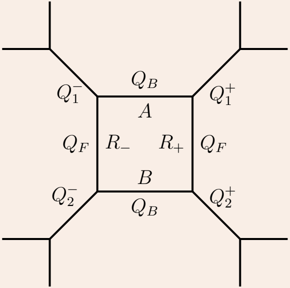
Topological strings

- Still nonlocal
- There is no tower of massive states
- Feel the geometry of the background manifold

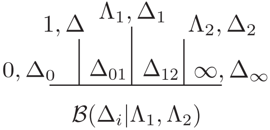
Toric manifolds

- $\mathbb{C}^N/U(1)^K$ quotient $z_i/(z_i \sim Q_i^A z_i)$
- Summarized by the toric diagram $\sum_i Q_i^A x_i = 0 \cap x_i \geq 0$
- Computed by the same recipe as the Feynman rules!

$$Z_{\text{top strings}} = \sum_{\text{edges}} \prod \text{propagators} \times \text{vertices}$$



Geometry of the toric three-fold encodes the data of the conformal blocks:



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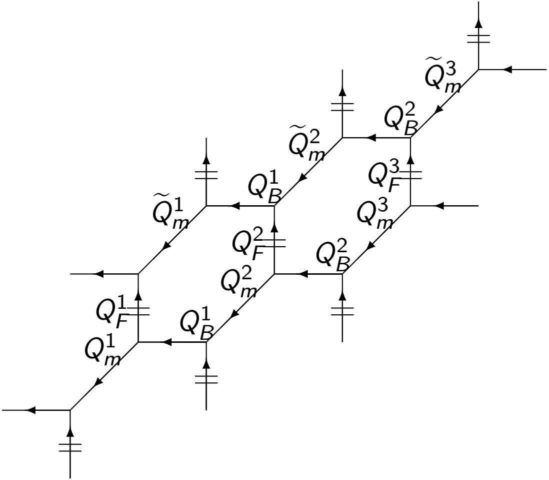
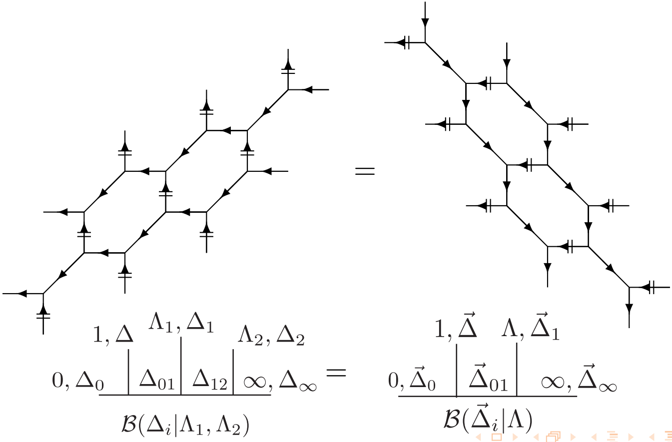


$\Lambda_1, \Lambda_2 \to Q_B^1, Q_B^2$
$\Delta_{01}, \Delta_{12} \to Q_F^1, Q_F^2$
$\Delta_i \to Q_m^i, \widetilde{Q}_m^i$

Spectral duality

- **Gauge theory:** In 5d instantons are ~~quasi~~particles, as are the gauge bosons. There is a symmetry between gauge bosons and instantons!
- **Topological strings:** rotation of toric diagram.
- q-**deformed CFT:** relation between different conformal blocks!



Conclusions

- There is an intricate web of dualities connecting $2d$ CFT, $4d$ & $5d$ gauge theories and topological strings.
- Spectral duality is natural in topological strings and in gauge theory.
- It gives new surprising relations for conformal blocks in $2d$ CFT.

Thank you!