Nucleon Sigma Terms from Lattice QCD

Lukas Varnhorst

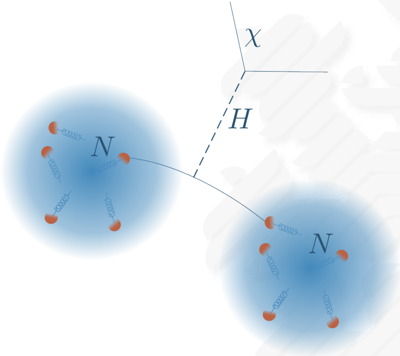
Bergische Universität Wuppertal
FB C - Department of Physics

Introduction

The nucleon-sigma-terms are of significant interest for dark-matter searches, as they determine the coupling of several dark matter candidates to hadronic matter.

The nucleon sigma terms are defined as

$$\sigma_{\pi N} = m_{ud} \langle N \mid \bar{u}u + \bar{d}d \mid N \rangle$$
$$\sigma_{\pi N} = 2m_s \langle N \mid \bar{s}s \mid N \rangle$$



Via the Feynmann-Hellman-Theorem they can be expressed as

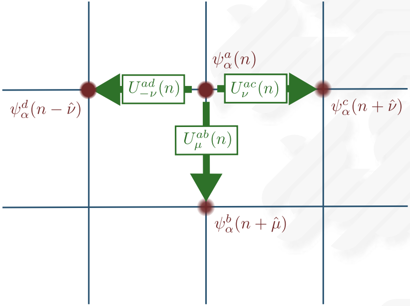
$$\sigma_{\pi N} = m_{ud} \frac{\partial M_N}{\partial m_{ud}} \quad \text{and} \quad \sigma_{\bar{s}sN} = 2m_{ud} \frac{\partial M_N}{\partial m_{ud}}$$

Lattice QCD

Lattice QCD regularizes QCD by replacing continuous space-time by a discrete lattice with lattice spacing a. Using a finite volume one can reduce the infinite number of degrees of freedom to a finite (but huge) number.

The fermionic degrees of freedom $\bar{\psi}(n)$ and $\psi(n)$ are defined only on the points of the lattice.

The gauge fields are defined as links $U_\mu(n)$ between the lattice points to exactly fulfill gauge invariance.



Lattice QCD

The task is to calculate the path integral

$$\langle A\rangle = \frac{1}{Z}\int \mathcal{D}U\, A \,\det M[U] \exp\left(-S_g[U]\right).$$

To achieve this one has to uses monte-carlo techniques. More specifically one sets up a Markov-Chain with equilibrium distribution $\det M[U] \exp\left(-S_g[U]\right)$. Using this Markov-process one can write

$$\langle A\rangle = \frac{1}{N}\sum_{i=1}^{N} A[U_i] + \mathcal{O}\left(\sqrt{\frac{2\tau+1}{N}}\right)$$

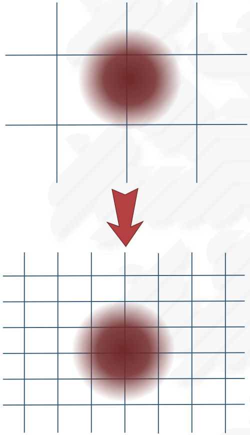
Lattice QCD

To make a lattice result exact one has to extrapolate it to zero lattice spacing.

We use a tree-level improved Symmanzik gauge action and a tree-level improved clover Wilson fermion action with $N_f = 1+1+1+1$ and three levels of HEX smearing.

This lattice action is supposed to have cut-off effects of order $\alpha_s a$. Often a^2 effects are dominant.

We used simulations at 4 different lattice spacing and extrapolated results to the continuum. This extrapolation has been performed both with $\alpha_s a$ and a^2 to estimate the systematic uncertainty.



Extracting hadron masses

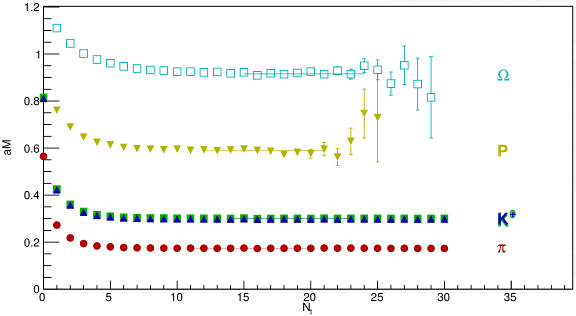
Suppose $\bar{O}$ and O are operators which create and annihilate a hadron H. The the correlation function between the two operators behaves as:

$$C(t) = \langle \bar{O}(t) O(0) \rangle = \sum_k \langle 0 \mid O \mid k \rangle \langle k \mid O^\dagger \mid 0 \rangle e^{-t \Delta E_k}$$

We can define a effective mass via

$$m_{\text{eff}}(t + \frac{1}{2}) = \ln \frac{C(t)}{C(t+1)}$$

and search for a plateau.



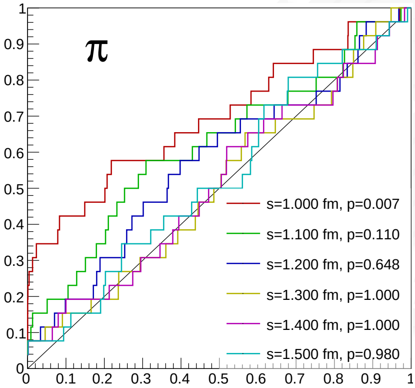
Extracting hadron masses

Even better: Fit correlation functions with

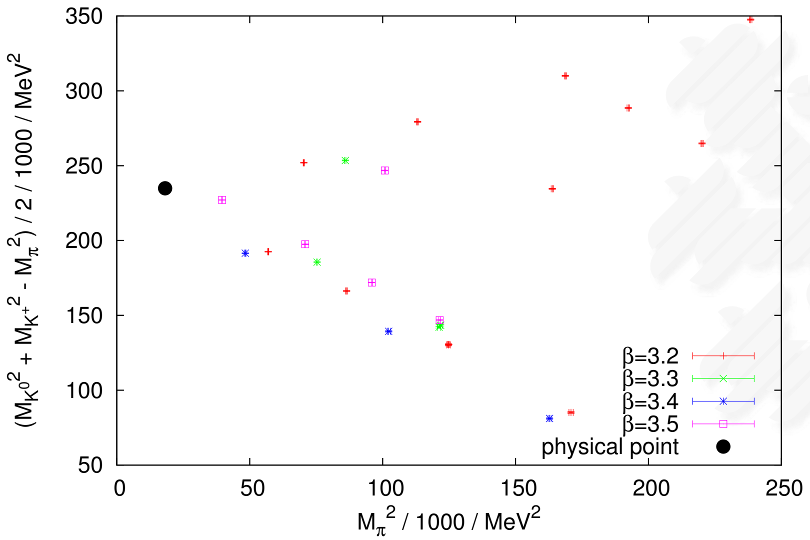
$$C(t) = \begin{cases} A\cosh(-m(t - N_t/2)) & \text{for mesons} \\ A\sinh(-m(t - N_t/2)) & \text{for baryons} \end{cases}$$

The question is: From which t_{min} on should the fit start?

We have many ensembles: We can make a Kolmogorov-Smirnov-Test to check wether the χ^2 values are properly distributed.



The landscape



Parameterizing the nucleon mass

We expect the nucleon mass to depend on the following quantities:

i. The lattice spacing a.
ii. The average light quark mass $m_{ud} \propto M_\pi^2$.
iii. The strange quark mass $m_s \propto M_{\bar{s}s} = \frac{1}{2}(M_{K^0}^2 + M_{K^+}^2 - M_\pi^2)$.
iv. The isospin splitting $M_{K^0}^2 - M_{K^+}^2$
v. The finite-volume via

$$\frac{M(L)}{M(\infty)} = 1 + \sqrt{\frac{M_\pi}{L^3}} e^{-M_\pi L} = 1 + F(L)$$

To leading order the nucleon mass can be fitted with the ansatz

$$M_N = \Big(c_0 + c_1(M_\pi^2 - M_\pi^{2,(\Phi)}) + c_2(M_{\bar{s}s}^2 - M_{\bar{s}s}^{2,(\Phi)}) + \\ + c_3(M_{K^+}^2 - M_{K^0}^2) + c_4 C(\beta)\Big)\left(1 + c_5 F(L)\right)$$

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to estimate systematic uncertainties we want to vary the fit function:

i. Continuum extrapolation may show an $C(\beta) = a^2(\beta)$ or an $C(\beta) = \alpha_s(\beta)a(\beta)$ behavior.
ii. The next-to-leading term in M_π^2 can be assumed to be M_π^4 (Taylor) or M_π^3 (χPT).
iii. Finite volume effects can be parametrized via $F(L) = \sqrt{\frac{M_\pi}{L^3}} e^{-M_\pi L}$ or via $F(L) = e^{-M_\pi L}$
iv. There might be higher order contribution in the $M_{\bar{s}s}$ and $M_{K^0}^2 - M_{K^+}^2$ direction.

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Our strategy is as follows:

Whenever there are two or more obvious choices one perform the entire analysis with each of the choices.

If there is only one obvious choice, then we add the corresponding term T in different ways:

$$\begin{aligned} M_N &\longrightarrow M_N + cT \\ M_N &\longrightarrow M_N(1 + cT) \\ M_N &\longrightarrow M_N/(1 - cT) \end{aligned}$$

The same techniques where also used for the scale setting.

Parameterizing the nucleon mass

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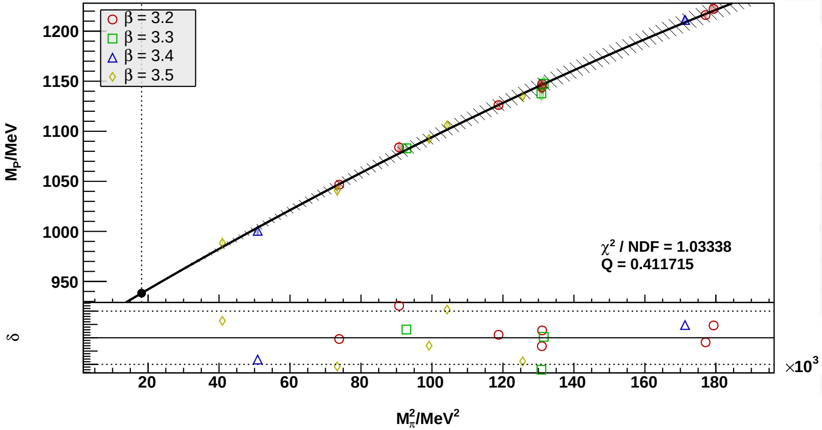

Parameterizing the nucleon mass

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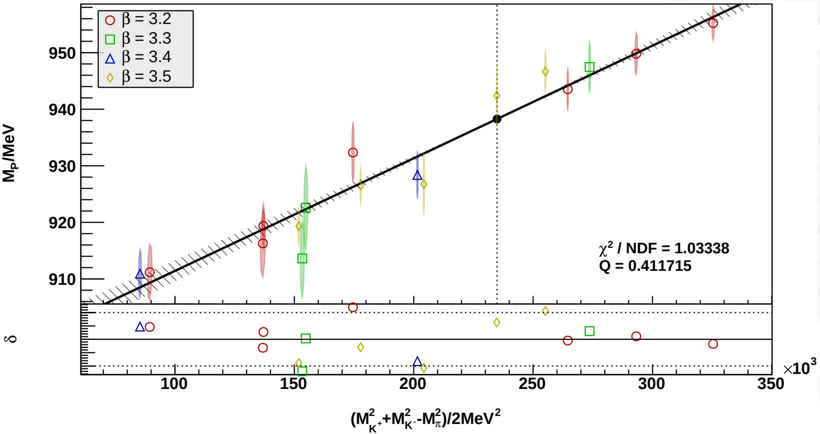

Towards the sigma term I

Using leading order χPT one may extract the sigma terms via

$$\sigma_{\pi N} = m_{ud} \frac{\partial M_N}{\partial m_{ud}} \approx M_\pi^2 \frac{\partial M_N}{\partial M_\pi^2}$$
$$\sigma_{\bar{s}sN} = 2m_{ud} \frac{\partial M_N}{\partial m_{ud}} \approx 2M_{\bar{s}s}^2 \frac{\partial M_N}{\partial M_{\bar{s}s}^2}$$

To make the analysis exact one has to determine the following matrix:

$$J = \begin{pmatrix} \frac{m_{ud}}{M_\pi^2} \frac{\partial M_\pi^2}{\partial m_{ud}}\Big|_{m_s} & \frac{m_{ud}}{M_{\bar{s}s}^2} \frac{\partial M_{\bar{s}s}^2}{\partial m_{ud}}\Big|_{m_s} \\ \frac{m_s}{M_\pi^2} \frac{\partial M_\pi^2}{\partial m_s}\Big|_{m_{ud}} & \frac{m_s}{M_{\bar{s}s}^2} \frac{\partial M_{\bar{s}s}^2}{\partial m_s}\Big|_{m_{ud}} \end{pmatrix}$$

It is easier to determine the inverse matrix via

$$(J^{-1})_{ij} = \frac{\partial m_i(M_\pi^2, M_{\bar{s}s}^2)}{\partial M_j^2}\Bigg|_{M_k^2}$$

The quark masses

What is left is to determine the quark masses as function of the meson masses.

Quark masses are much more difficult then hadron masses. On the lattice quark masses can be defined in several ways:

i. **The VWI quark masses** The bare quark masses of the Wilson quark action are subject to additive renormalization:

$$am^W = am^B - am^c$$

From this quantity one can construct the VWI quark mass:

$$m_j^{VWI} = \frac{1}{Z_S} m_j^W \left(1 - \frac{1}{2} b_S a m_j^W - \bar{b}_S a \,\mathrm{tr}\, M + \mathcal{O}(a^2)\right)$$

Where Z_S is the renormalization factor of the scalar current.

The quark masses

ii. **The AWI quark masses** Form the axial ward identity one can define so called PCAC quark masses:

$$\langle \partial_\mu A_\mu^{(j,k)} \rangle_J = (m_j + m_k) \langle P^{(j,k)} \rangle_J$$

From these one can define the AWI quark masses:

$$m_j^{AWI} = \frac{Z_A}{Z_P} m_j^{PCAC} \left(1 + (b_A - b_P) a m_j^W + (\bar{b}_A - \bar{b}_P) a \operatorname{tr} M + \mathcal{O}(a^2)\right)$$

The AWI quark masses renormalized only multiplicatively with Z_A and Z_P are the renormalization factors of the axial and the pseudoscalar current.

Both definitions can be combined to the ratio-difference quark mass m^{rd}

Fitting quark masses

The quark masses extracted via the ratio difference method still renormalizes multiplicatively. However in quantities of the form

$$\frac{m_{ud}}{M_\pi^2}\frac{\partial M_\pi^2}{\partial m_{ud}}\bigg|_{m_s}, \text{etc.}$$

the renormalization factors cancel out. It is therefore enough to determine the ratio

$$R_q = \frac{\hat{m}_q^{rd}}{\hat{m}_q^{rd,(\Phi)}} = \frac{m_q^{rd}}{m_q^{rd,(\Phi)}}$$

where $\hat{m}_q^{rd}$ is the renormalized quark mass. From this one can get

$$(J^{-1})_{ij} = \frac{\partial R_i(M_\pi^2, M_{\bar{s}s}^2)}{\partial M_j^2}\bigg|_{M_k^2}$$

Fitting quark masses

One can now make an expansion of the quark mass around the physical point much like for the nucleon mass.

$$R_q = \frac{m_q^{rd}}{m_q^{rd,(\Phi)}(\beta)} = \frac{1}{m_q^{rd,(\Phi)}(\beta)} \Big(1 + c_1(M_\pi^2 - M_\pi^{2,(\Phi)}) + \\ + c_2(M_{\bar{s}s}^2 - M_{\bar{s}s}^{2,(\Phi)}) + c_3(M_{K^+}^2 - M_{K^0}^2)\Big)(1 + c_5 F(L))$$

The value of the unrenormalized quark mass at the physical point at a given lattice spacing is now an additional beta depended fit parameter $m_q^{rd,(\Phi)}(\beta)$.

Fitting quark masses

The contributions of higher order terms are estimated using the following considerations

i. The next-to-leading term in M_π can be assumed to be either taken from a Taylor expanion or from χPT.

ii. Finite volume effects can be parametrized via $F(L) = \sqrt{\frac{M_\pi}{L^3}} e^{-M_\pi L}$ or via $F(L) = e^{-M_\pi L}$

iii. There might by higher order contribution in the $M_{\bar{s}s}$ and $M^2_{K^0} - M^2_{K^+}$ direction.

Towards the sigma term II

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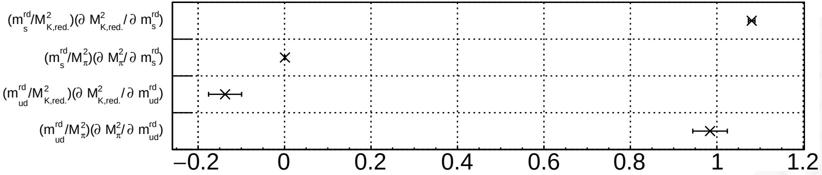
From thees fits we can deduce the entries of J (errors purely statistical):



Now we can construct the sigma terms:

$$\sigma_{\pi N} = \frac{m_{ud}}{M_\pi^2}\frac{\partial M_\pi^2}{\partial m_{ud}}\bigg|_{m_s} M_\pi^2 \frac{\partial M_N}{\partial M_\pi^2} + \frac{m_{ud}}{M_{\bar{s}s}^2}\frac{\partial M_{\bar{s}s}^2}{\partial m_{ud}}\bigg|_{m_s} M_{\bar{s}s}^2 \frac{\partial M_N}{\partial M_{\bar{s}s}^2}$$

$$\sigma_{\bar{s}sN}/2 = \frac{m_s}{M_\pi^2}\frac{\partial M_\pi^2}{\partial m_s}\bigg|_{m_{ud}} M_\pi^2 \frac{\partial M_N}{\partial M_\pi^2} + \frac{m_s}{M_{\bar{s}s}^2}\frac{\partial M_{\bar{s}s}^2}{\partial m_s}\bigg|_{m_{ud}} M_{\bar{s}s}^2 \frac{\partial M_N}{\partial M_{\bar{s}s}^2}$$

Uncertaintys

To estimate the systematic uncertainties we cary out a set of analyses, each of which is valid. We varied:

- The details of the fit functions as described above
- We applied three different pion mass cuts.
- We performed the scale setting using the Ω-mass and the nucleon mass itself.

Altogether we have 864 different analyses. We make a histogram of thees analyses and determine the spread.

For the statistical error we have performed a bootstrap analysis.

The total error is:

$$\sigma = \sqrt{\sigma_{\text{stat}}^2 + \sigma_{\text{syst.}}^2}$$

Results

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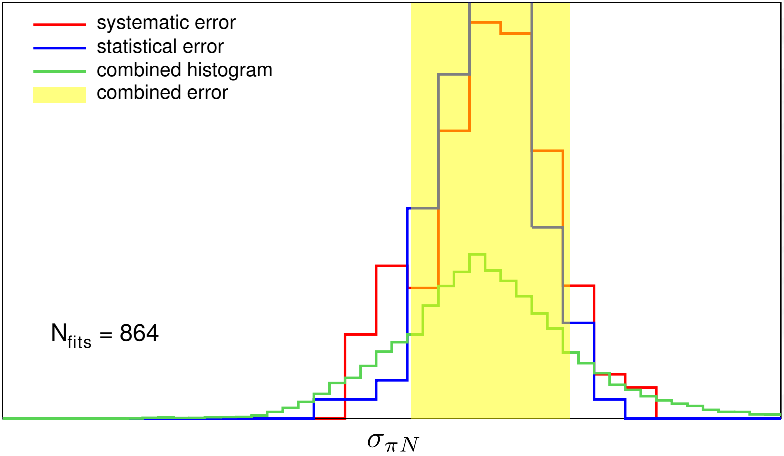

Results

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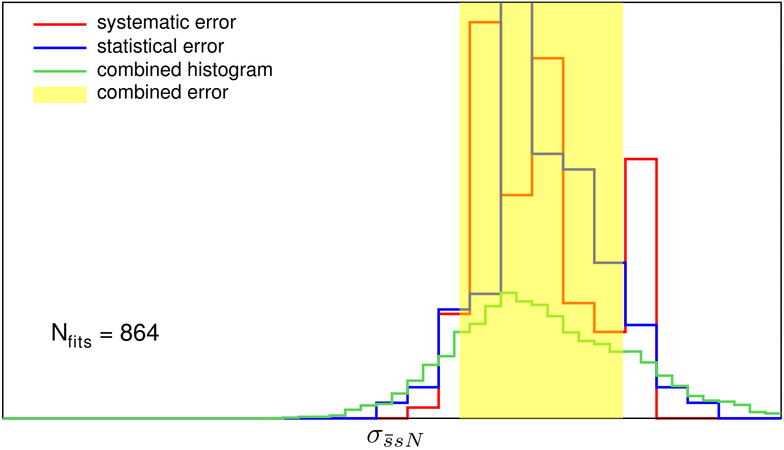

Conclusion

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- Sigma terms can be calculated on the lattice via Feynman-Hellman theorem.
- Calculating the mass dependence of the proton is done routinely.
- Exact functional form becomes more important once derivatives are calculated.
- One can deal with the quark masses without explicitly renormalizing them.
- Systematic uncertaintys are under conntrol.