Kantowski-Sachs universes sourced by a Skyrme field

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Introduction

The Kantowski-Sachs metric describe spatially homogeneous anisotropic spacetimes with a four-dimensional isometry group whose three-dimensional subgroup acts multiply transitively on two-dimensional spherically symmetric surfaces.[1]

$$ds^2 = -dt^2 + A(t)^2\, dr^2 + B(t)^2\left[d\theta^2 + \sin\theta^2 d\phi^2\right].$$

The global structure of these models was described by Collins, who was also the first who analyzed the model as a two-dimensional dynamical system for the case of perfect fluid with vanishing cosmological constant.[2]

Motivated by a series of recent papers concerning the analyses of self-gravitating Skyrme fields in cosmology and Kantowski-Sachs spacetimes[3], we consider Kantowski-Sachs cosmological models sourced by a Skyrme field and a cosmological constant in the framework of General Relativity.

Einstein-Skyrme system

The Skyrme model is a generalized nonlinear sigma model. Although not involving quarks, it can be regarded as an approximate, low energy effective theory of QCD, whose topological soliton solutions can be interpreted as baryons (Skyrmions).

The Einstein-Skyrme equations read:

$$G_{\mu\nu} + \Lambda\, g_{\mu\nu} = 8\pi G\, T^S_{\mu\nu}, \qquad (1)$$

$$\nabla^\mu R_\mu + \frac{\lambda}{4}\nabla^\mu\left[R^\nu, F_{\mu\nu}\right] = 0 \qquad (2)$$

where $T^S_{\mu\nu}$ is the energy-momentum tensor for the Skyrme field, $G_{\mu\nu}$ is the Einstein tensor, Λ is the cosmological constant, G is the Newton constant, K (see below) and λ are coupling constants, $F_{\mu\nu}$ and R_μ are, respectively, the field strength and a $\mathbf{su}(2)$-valued current for the Skyrme field U which takes values on a specific target manifold, the Lie group $SU(2)$.

In what follows, beside the choice of Kantowski-Sachs spacetimes, we consider the particular case of a constant radial profile function $\alpha=\pi/2$ so that Eq.(2), reduced to scalar equation by the hedgehog ansatz, is identically solved.

Eq.(1) can be further manipulated and written in terms of propagation equations for the usual volume expansion scalar θ, the shear scalar $\sigma^2=(1/2)\sigma^{\mu\nu}\sigma_{\mu\nu}$ (where $\sigma^{\mu\nu}$ is the shear tensor) and the 3-curvature scalar ${}^{(3)}R$.

$$\dot\theta + \frac{1}{3}\theta^2 + 2\sigma^2 = \Lambda - \frac{k\lambda\,{}^{(3)}R^2}{8} \qquad (3)$$

$$\dot\sigma + \theta\sigma - \frac{1}{2\sqrt3}\,{}^{(3)}R = -\frac{k\,{}^{(3)}R}{4\sqrt3}\left(\lambda\,{}^{(3)}R + 2\right) \qquad (4)$$

$${}^{(3)}\dot R + \frac{2}{3}\theta\,{}^{(3)}R - \frac{2}{\sqrt3}\,{}^{(3)}R\sigma = 0 \qquad (5)$$

$${}^{(3)}R + \frac{2}{3}\theta^2 - 2\sigma^2 = 2\Lambda + k\,{}^{(3)}R\left(1 + \frac{\lambda\,{}^{(3)}R}{4}\right). \qquad (6)$$

In what follows $8GK\to k$ and we will consider $0<k<1$.

Dynamical system

Introducing the normalization function $D \equiv \sqrt{\frac{1}{9}\theta^2 + \frac{1}{6}\,{}^{(3)}R}$ it is possible to define new dimensionless variables

$$Q = \frac{\theta}{3D},\quad \Sigma^2 = \frac{\sigma^2}{3D^2},\quad \Omega_\Lambda = \frac{\Lambda}{3D^2},\quad \Omega_k = \frac{{}^{(3)}R}{6D^2},\quad \Omega_s = \frac{k\lambda\,{}^{(3)}R^2}{24D^2}$$

This allows to construct a compact state space since the constraint in Eq.(6) becomes $Q^2 + \Omega_k = 1,\ \ and\ \ \Omega_\Lambda + \Sigma^2 + k(1-Q^2) + \Omega_s = 1.$

Differentiating with respect to the new time variable $' \equiv (1/D)d/dt$ and making use of the above constraint, the system can be eventually reduced to a three-dimensional autonomous dynamical system in the new variables $Q, \Sigma, \Omega_\Lambda$:

$$\begin{aligned}
Q' &= (Q^2-1)(1-k(1-Q^2)+Q\Sigma+\Sigma^2-2\Omega_\Lambda)\\
\Sigma' &= k(1-Q^2)(1-Q\Sigma) - (1-\Sigma^2)\left[1+Q(Q+\Sigma)\right] + 2(1-Q\Sigma)\Omega_\Lambda\\
\Omega'_\Lambda &= 2\left[Q\left(2-k(1-Q^2)\right)+\Sigma(Q+\Sigma-1) - 2\Omega_\Lambda\right]\Omega_\Lambda
\end{aligned}$$

with compact phase space

$$\mathcal{S} = \{(Q,\Sigma,\Omega_\Lambda)\in\mathbb{R}|-1\le Q\le 1, -1\le\Sigma\le 1, 0\le\Omega_\Lambda\le 1, 0\le 1-\Omega_\Lambda-\Sigma^2-k(1-Q^2)\le 1\}$$

The system admits **six stationary points** listed below.

Point	Q	Σ	Ω_Λ	Ω_K	Ω_S	Stability	q	θ
$\mathcal{A}$	-1	-1	0	0	0	Stable (attractor)	2	$\sim t^{-1}$
$\mathcal{B}$	-1	0	1	0	0	Unstable (repeller)	-1	$\sim$ const.
$\mathcal{C}$	-1	1	0	0	0	Unstable (saddle)	2	$\sim t^{-1}$
$\mathcal{F}$	1	-1	0	0	0	Unstable (saddle)	2	$\sim t^{-1}$
$\mathcal{G}$	1	0	1	0	0	Stable (attractor)	-1	$\sim$ const.
$\mathcal{H}$	1	1	0	0	0	Unstable (repeller)	2	$\sim t^{-1}$

It also displays a **normally hyperbolic equilibrium set** in the $Q= \Sigma$ plane, defined by:

$$-\sqrt{\frac{1-k}{4-k}} \le Q \le \sqrt{\frac{1-k}{4-k}},\qquad \Omega_\Lambda = \frac{1}{2}(1+2Q^2+k(Q^2-1))$$

and **three invariant submanifolds** characterized by $Q=1$, $Q=-1$ and $\Omega_\Lambda=0$ respectively.

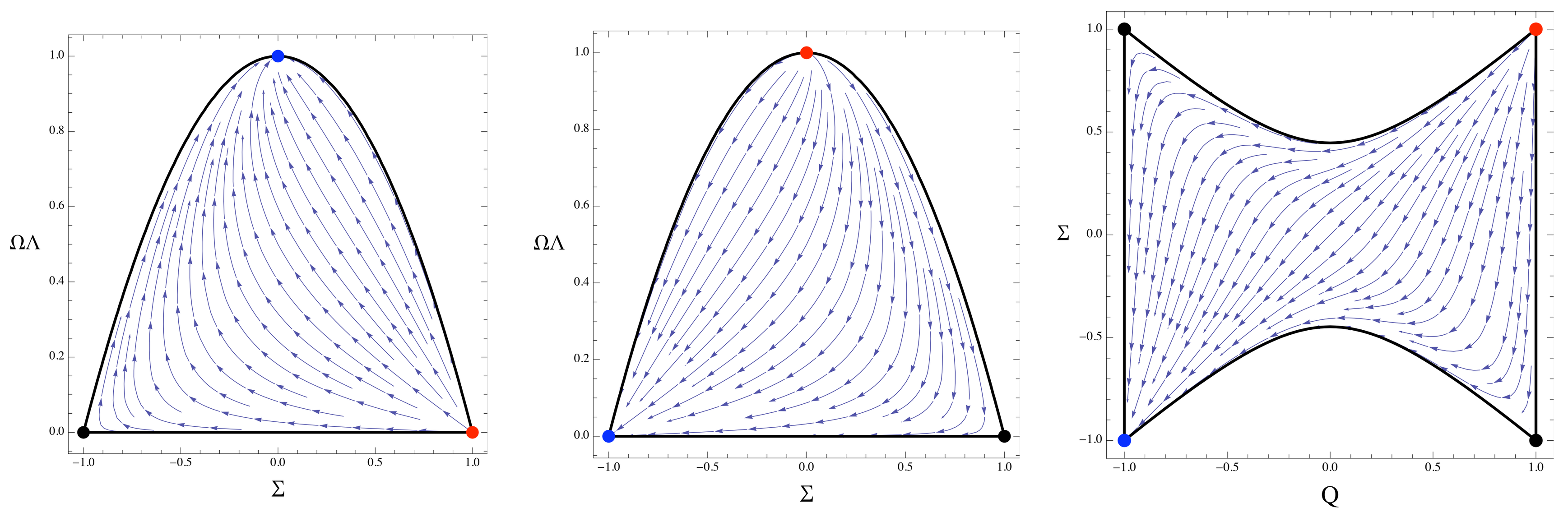

Figure 1: Invariant submanifold $Q = 1$.

Figure 2: Invariant submanifold $Q = -1$.

Figure 3: Invariant submanifold $\Omega_\Lambda = 0$.

Discussion

The numerical value of the deceleration parameter q and the analytic behavior of the expansion scalar θ at each fixed point can be easily evaluated. Two classes of solution are obtained.

The stationary points $\mathcal{A}$, $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{H}$ are characterized by $Q^2=1$ and $\Sigma^2=1$; this allows to solve in terms of both scale factors A and B to obtain either

$$B\sim\text{const.}\quad\text{and}\quad A\sim t,\quad\text{or}\quad B\sim t^{2/3}\quad\text{and}\quad A\sim t^{-1/3},$$

depending on the sign of Σ. These solutions are anisotropic and display Kasner-like behaviours.

Since the solutions represented by the stationary points $\mathcal{B}$ and $\mathcal{G}$ have vanishing shear they are said to undergo isotropization which, in this context, means that the two scale factors are characterized by the same functional dependence on time. They are driven by the cosmological constant, the sign of the exponent depending on the sign of Q:

$$B\sim A\sim e^{\pm\sqrt{\frac{\Lambda}{3}}t},$$

thus these solutions are said de Sitter-like solutions. Analogously, for the equilibrium set the acceleration parameter is always $q=-1$.

A **bounce** in one of the scale factors, say A, is said to occurs at time t^* if and only if $y(t^*)=0$ and $dy/dt\,(t^*)>0$, where $y=(1/A)\,dA/dt$.

It can be easily shown that, in this model, a bounce in the scale factor B is impossible while a bounce in the scale factor A requires the violation of the Strong Energy Condition of the total matter-energy content.

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