Non-Abelian vortices with a twist

Péter Forgács[1,2], Árpád Lukács[1,3], Fidel A. Schaposnik[4]

[1] Field Theory Research Group, Department of Theoretical Physics,
MTA Wigner RCP Institute for Particle and Nuclear Physics, H1525 Budapest, POB 49
[2] LMPT, CNRS-UMR 6083, Université de Tours, Parc de Grandmont, 37200 Tours, France
[3] Aristotle University of Thessaloniki, Greece

lukacs.arpad@wigner.mta.hu, www.rmki.kfki.hu/~arpi, arXiv:1504.05129 [hep-th]

Phys.Rev. **D91** (2015) 12, 125001

1. Introduction

1.1 Vortices

Abrikosov, Nielsen and Olesen:

$$\phi = f(r)\exp(in\vartheta)\,,$$

$f(0)=0$, $f(r\to\infty)=1$, topology: winding number

1.2 The theory considered

Bosonic sector of $N=2$ supersymmetric $SU(2)\times U(1)$ gauge theory, $SU(2)$ flavor symmetry.
$S=\int \mathrm{d}^4x\mathcal{L}$,

$$\mathcal{L} = -\frac{1}{4g_1^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4g_2^2}G^a_{\mu\nu}G^{\mu\nu a} + \mathrm{Tr}(D_\mu\Phi)^\dagger D^\mu\Phi - (V_1+V_2)\,,$$

where $D_\mu\Phi = (\partial_\mu - iA_\mu\sigma^0/2 - iC^a_\mu\sigma^a/2)\Phi$
Potential

$$V_1 = \frac{\lambda_1}{8}(\mathrm{Tr}\,\Phi^\dagger\Phi - 2\xi)^2\,,\quad V_2 = \frac{\lambda_2}{8}(\mathrm{Tr}\,\Phi^\dagger\sigma^a\Phi)^2$$

Properties of this theory:

- scalar sector of a supersymmetric theory
- possesses many localized solutions (strings, etc.)
- For $\lambda_i = g_i^2$: SUSY
- For $\lambda_i = g_i^2$: BPS first order equations

Spontaneous symmetry breaking

Let

$$\Phi = \begin{pmatrix}\phi_1 & \psi_1\\ \phi_2 & \psi_2\end{pmatrix}$$

with this notation:

$$V_1 = \frac{\lambda_1}{8}(\phi^\dagger\phi + \psi^\dagger\psi - 2\xi)^2\,,\quad V_2 = \frac{\lambda_2}{8}\left[(\phi^\dagger\phi - \psi^\dagger\phi)^2 + 4|\psi^\dagger\phi|^2\right]\,,$$

i.e., vacuum: both ϕ,ψ normalized to ξ and orthogonal
Symmetry breaking pattern:

$$U(1)\times SU(2)\times SU(2)_{\text{global}} \to SU(2)_{\text{CF}}$$

where $SU(2)_{CF}$ preserves the VEV,
E.g., choosing $\langle\Phi\rangle = \xi\mathbb{1}$: $SU(2)_{CF}$ acts as $\Phi\to V\Phi V^\dagger$

Color-flavor locking: gauge and color symmetry both broken spontaneously, $SU(2)_{\text{CF}}$ remains unbroken

Topology permits vortex solutions

2. Vortices in the plane

Rotationally symmetric Ansatz:

$$\Phi(x^i) = \begin{pmatrix}\phi_1(r)e^{in_1\vartheta} & \psi_1(r)e^{in_1\vartheta}\\ \phi_2(r)e^{im_1\vartheta} & \psi_2(r)e^{im_1\vartheta}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{iN\vartheta}\end{pmatrix} = \Phi_0(x^i)e^{i\mathbf{N}\vartheta}\,,$$

and $\partial_\vartheta\{A_\mu, C^a_\mu\} = 0$.
Symmetries:

- gauge: $\Phi\to U(x)\Phi$: if $U\partial_\vartheta U^\dagger$ ϑ-indept
- flavor: $\Phi\to\Phi V$ if $[N,V]=0$

An example:

$$\Phi = \begin{pmatrix}\phi_1(r) & \\ & \psi_2(r)\end{pmatrix}\,,\qquad \begin{aligned}A_\vartheta &= a(r)\\ C^3_\vartheta &= c_3(r)\end{aligned}$$

with real radial functions
Further solutions generated: orientational normal modes

$$\Phi\to V\Phi V^\dagger\,,\quad V\in SU(2)$$

explicitly:

$$\Phi = \chi_+\mathbb{1} + \chi_- n^a\sigma^a\,,\quad c_a = n^a\tilde{c}_3$$

where $\chi_\pm = (\phi_{1D}+\phi_{2D})/2$.
(Hanany, Tong 2003; Auzzi etal. 2003, Auzzi, Shifman, Yung 2006)

3. Twisted strings

3.1 Nontrivial dimensional reduction

Straight string: translation invariance along axis z:

$$\Phi(x^\mu) = \Phi(x^i)\exp\left(\frac{i}{2}M\omega_\alpha x^\alpha\right)\,,$$
$$A_\mu(x^\nu) = (A_i(x^j), A_\alpha(x^j))\,,$$
$$C^a_\mu(x^\nu) = (C^a_i(x^j), C^a_\alpha(x^j))\,,$$

Decoupling

$$\omega^2 = -\omega_\alpha\omega^\alpha = 0$$

ensures that the equations for $\Phi(x^i)$, C^a_i, A_i are unchanged $(i=1,2)$

$$A_\alpha = \omega_\alpha A\,,\quad C^a_\alpha = \omega_\alpha C^a$$

The out-of-plane components satisfy a Gauss-constraint

Solutions equivalent to solving mass deformed theory

- adjoint scalars: out-of-plane gauge field components
- mass matrix – twisting matrix

(Collie, Eto etal., Gorsky etal.)
Gauss constraint for out-of-plane fields (Eto etal. 2011):

$$D_i^2\left[\frac{1}{g_1^2}A\sigma^0 + \frac{1}{g_2^2}C^a\sigma^a\right] = -\Phi M\Phi^\dagger + \frac{1}{2}\{C,\Phi\Phi^\dagger\}\,,$$

where $C = A + C^a\sigma^a$, planar vortex fields $\to$ background.
Physical quantities: energy:

$$E = E_{\text{BPS}} + \frac{\omega_0^2+\omega_3^3}{4}\int\mathrm{d}^2xQ\,,$$

where $E_{\text{BPS}} = 2\pi|n_1+m_1+N|$ (no. of flux quanta), momentum:

$$P = \frac{1}{2}\omega_0\omega_3\int\mathrm{d}^2xQ$$
$$Q = \mathrm{Tr}\left[\Phi^\dagger(\Phi M - C\Phi)M\right]\,,$$

and angular momentum

$$J = \int\mathrm{d}^2x\omega_0\frac{1}{2}\mathrm{Tr}\left[\Phi^\dagger\Phi(NM+MN) - 2\Phi^\dagger C\Phi N\right]$$

where $\partial_\vartheta\Phi = iN\Phi$.

3.2 Symmetries

The Ansatz

$$\Phi(x^\mu) = \Phi(x^i)\exp\left(\frac{i}{2}M\omega_\alpha x^\alpha\right)$$

restricts the symmetries:

- gauge: $\Phi\to U(x)\Phi$: if $U\partial_\alpha U^\dagger$ x^α-independent
- flavor: $\Phi\to\Phi V$: if $[V,M]=0$
  $V = \exp(iM\delta)\,,$

Noether current:

$$J_\mu = m^{\hat a}K^{\hat a}_\mu\,,\quad J_\alpha = \omega_\alpha Q\,,$$

where $M = m^0 + m^a\sigma^a = m^{\hat a}\sigma^{\hat a}$ Here,

$$K^{\hat a}_\mu = \frac{i}{2}\mathrm{Tr}\left[D_\mu\Phi\sigma^{\hat a}\Phi^\dagger - \Phi\sigma^{\hat a}D_\mu\Phi^\dagger\right]\,,$$

is the Noether current of flavor symmetry ($K^0_\mu = -J^0_\mu$).

Rotation and translation in z

Symmetries represented nontrivially: compensation

$$\Phi(x) = U(x)\Phi(x')V$$

(Forgács, Manton, 1980)

Flavor transformations restricted by the Ansatz

- ϑ dependence: $[V,N]=0$
- x^α dependence: $[V,M]=0$

Simultaneously: if $[M,N]=0$

Compensate rotations with internal symmetry? If

$$[M,N]\neq 0$$

rotational symmetry is lost. Cross sections still symmetric.

3.3 The elementary vortex

3.4 Twisting the elementary vortex

Reminder: elementary vortex: $\Phi = V\Phi_D V^\dagger$:

$$\Phi = \chi_+\mathbb{1} + \chi_- n^a\sigma^a\,,\quad c_a = n^a\tilde{c}_3$$

where $\chi_\pm = (\phi_{1D}+\phi_{2D})/2$.

Ansatz for out-of-plane fields: let $M = m^0\sigma^0 + m^a\sigma^a$,

$$A = m^0\,,\quad C^a = (mn)n^a + \tilde{m}^a C(r)\,,\quad \tilde{m}^a = m^a - (mn)n^a\,,$$

yielding one equation

$$\frac{1}{r}(rC')' - \frac{c_3^2}{r^2}C = g_2^2\left[\chi_+^2(C-1) + \chi_-^2(C+1)\right]\,.$$

Note: m^0 and parallel part to n^a: pure gauge

$$Q = \left(m^2 - (nm)^2\right)\left(C(r)(\chi_-^2 - \chi_+^2) + (\chi_-^2 + \chi_+^2)\right)\,,$$

(see also Collie, Eto etal.)

3.5 Coincident composite vortices

$$\Phi(x^i) = \begin{pmatrix}\phi_1(r) & \psi_1(r)e^{iN\vartheta}\\ \phi_2(r) & \psi_2(r)e^{iN\vartheta}\end{pmatrix}\,,$$
$$A_\vartheta = a(r)\,,$$
$$C^a_\vartheta = c_a(r)\,.$$

(Auzzi, Shifman, Yung, 2006)
Parameterization: $r\to\infty$

$$\phi_1\to\cos\alpha\,,\quad \phi_2\to\sin\alpha\,,$$
$$\psi_1\to -\sin\alpha\,,\ \psi_2\to\cos\alpha\,,$$
$$a\to N\,,\quad c_1\to -N\sin(2\alpha)\,,\quad c_3\sim -N\cos(2\alpha)\,.$$

Add twist introduce radial functions

$$M = m^{\hat a}\sigma^{\hat a}\,,\quad m^0 = -m^3 = \frac{s}{2}\,,\quad m^1 + im^2 = \frac{m}{2}e^{i\mu}\,,$$
$$A = sA_0 + mA_+e^{i(N\vartheta+\mu)} + mA_-e^{-i(N\vartheta+\mu)}\,,$$
$$C^a = sC^a_0 + mC^a_+e^{i(N\vartheta+\mu)} + mC^a_-e^{-i(N\vartheta+\mu)}\,,$$

3.5.1 Energy, momentum

$$M = m^{\hat a}\sigma^{\hat a}\,,\quad m^0 = -m^3 = \frac{s}{2}\,,\quad m^1 + im^2 = \frac{m}{2}e^{i\mu}\,,$$

Insert Ansatz:

$$\int\mathrm{d}^2xQ = s^2\int\mathrm{d}^2xQ_s + m^2\int\mathrm{d}^2xQ_m\,,$$

and

$$\int\mathrm{d}^2xJ = s^2\omega_0N\int\mathrm{d}^2xQ_s\,.$$

$s=0$: no total angular momentum

3.5.2 Numerical solutions

For $g_2=1$: $Q^{\text{tot}}_{m,s} = \int dx^2xQ_{m,s}$

g_1	α	Q_s^{tot}	Q_m^{tot}
0.4	0	0	6.283
	0.05	0.0249	6.222
	0.785398	3.118	3.591
0.77	0	0	6.507
	0.05	0.0243	6.261
	0.785398	2.966	3.591
2.33	0	0	6.222
	0.05	0.0232	6.211
	0.785398	2.683	3.586

$E_1 = 4\pi$. $\alpha=0$: an elementary vortex, s twist: gauge

3.5.3 Constraints, quadrature

A surprise: numerically $A_\pm = 0$

- Imposing it is consistent with the field equations

$$\triangle^{(N)}A_\pm = g_1^2(\eta A_\pm + \eta^{\bar a}C^{\bar a}_\pm - \frac{m}{4}\chi^0)$$

- Results in a constraint. The constraint and its derivatives give:

$$r\eta^3{C^1_\pm}' = -(c_3\eta + N\eta^3_-)C^1_\pm + (c_3\chi^1 + iN\eta^3_1\eta^3_2/\chi^2)/4\,,$$

- and for the remaining fields:

$$\eta^1C^1_\pm + \eta^3C^3_\pm - \frac{1}{4}\chi^0 = 0\,,$$
$$\chi^1C^1_\pm \pm \chi^2C^2_\pm + \chi^3C^3_\pm = 0\,.$$

  where the coefficient functions contain only the background
- A remainder of rotational symmetry
- Reduces the solution of the Gauss equations to **quadrature**.

4. Conclusions

- Static vortices well known
  - in the Abelian Higgs model
  - in non-Abelian gauge theories
  - BPS case (SUSY)
- With multiple fields, twisted strings also possible
- Known for the elementary vortex
- Interesting properties for the composite coincident vortex
  - Energy difference sometimes very small
  - Momentum in the direction of the string axis
  - Rotating and no net angular momentum case

References

- M. Shifman and A. Yung, *Supersymmetric solitons*, CUP, 2009. (see also references therein)
- A. Hanany and D. Tong, *JHEP* **0307**, 037 (2003)
- R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, *Nucl. Phys.* **B 673**, 187 (2003)
- B. Collie, *Dyonic non-Abelian vortices J. Phys.* **A42** (2009) 085404
- A. Gorsky, M. Shifman, and A. Yung, *Phys. Rev.* **D71**, 045010 (2005)
- M. Eto, T. Fujimori, M. Nitta, K. Ohashi, and N. Sakai, *Dynamics of non-Abelian vortices Phys. Rev.* **D84** (2011) 125030.
- R. Auzzi, M. Shifman, A. Yung, *Composite non-Abelian flux tubes in $\mathcal{N}=2$ sQCD Phys. Rev.* **D73**, 105012 (2006).
- M. Eto, Y. Hirono, M. Nitta, S. Yasui, *Vortices and Other Topological Solitons in Dense Quark Matter*, arXiv:1308.1535
- E. Abraham, *Charged semilocal vortices Nucl. Phys.* **B399** (1993) 197-210.
- P. Forgács, N.S. Manton, *Commun. Math. Phys.* **72** (1980) 15.
- P. Forgács, S. Reuillon, and M.S. Volkov, *Twisted superconducting semilocal strings*, *Nucl. Phys.***B 751** (2006) 390–418.