Mixing generator: angle and mass dependance

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Abstract
Mixing transformation in Quantum Field Theory (QFT) are not trivial, since they induce a vacuum structure given by a condensate of particle-antiparticle pairs. Here we show that it is possible to decompose the mixing generator in terms of a rotation transformed under two Bogoliubov transformations, responsible for mass shifts. This result can be useful for a better understanding of the dynamical mechanism underlying flavor mixing generation.

Introduction

Apart from the problem of the origin of mixing and of the small neutrino masses, difficulties arise already in the attempt to find a proper mathematical setting for the description of mixed particles in Quantum Field Theory (QFT). Recent results in the area of field mixing and oscillations, starting from Ref, [1], have shown that, realizing the unitary inequivalence of the mass and flavor representations, a consistent field theoretical treatment is possible, both for fermions and for bosons.

Fermion mixing[1]

Considering [2] two flavor fields ν_e, ν_μ, the mixing relations are

$$\nu_e(x) = \cos\theta\, \nu_1(x) + \sin\theta\, \nu_2(x)\,; \qquad \nu_\mu(x) = -\sin\theta\, \nu_1(x) + \cos\theta\, \nu_2(x)\,, \tag{1}$$

Here ν_e, ν_μ are the (Dirac) neutrino fields with definite flavors. ν_1, ν_2 are the (free) neutrino fields with definite masses m_1, m_2, respectively. θ is the mixing angle. The fields ν_1 and ν_2 are expanded as

$$\nu_i(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u^r_{\mathbf{k},i}(t)\alpha^r_{\mathbf{k},i} + v^r_{-\mathbf{k},i}(t)\beta^{r\dagger}_{-\mathbf{k},i} \right] e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad i = 1,2\,. \tag{2}$$

where $u^r_{\mathbf{k},i}(t) = e^{-i\omega_{k,i}t}u^r_{\mathbf{k},i}$ and $v^r_{\mathbf{k},i}(t) = e^{i\omega_{k,i}t}v^r_{\mathbf{k},i}$, with $\omega_{k,i} = \sqrt{\mathbf{k}^2 + m_i^2}$. The $\alpha^r_{\mathbf{k},i}$ and the $\beta^r_{\mathbf{k},i}$ $(r = 1,2)$, are the annihilation operators for the vacuum state $|0\rangle_{1,2} \equiv |0\rangle_1 \otimes |0\rangle_2$: $\alpha^r_{\mathbf{k},i}|0\rangle_{1,2} = \beta^r_{\mathbf{k},i}|0\rangle_{1,2} = 0$.
The anticommutation relations are the standard ones[3].
Eqs.(1) can be recast as [1]:

$$\nu_e^\alpha(x) = G_\theta^{-1}(t)\, \nu_1^\alpha(x)\, G_\theta(t)\,; \qquad \nu_\mu^\alpha(x) = G_\theta^{-1}(t)\, \nu_2^\alpha(x)\, G_\theta(t)\,, \tag{3}$$

where $G_\theta(t)$ is given by

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x) \right) \right]\,, \tag{4}$$

The action of the mixing generator on the vacuum $|0\rangle_{1,2}$ is non-trivial and we have (at finite volume V):

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t)\, |0\rangle_{1,2}\,. \tag{5}$$

where $|0(t)\rangle_{e,\mu}$ is the *flavor vacuum*, i.e. the vacuum for the flavor fields. Moreover, using the Gaussian decomposition for G_θ^{-1} [2]we obtain that in the infinite volume limit (for any t)

$$\lim_{V\to\infty} {}_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty} e^{\frac{V}{(2\pi)^3}\int d^3\mathbf{k}\, ln\, (1-\sin^2\theta\, |V_\mathbf{k}|^2)} = 0 \tag{6}$$

Eq.(6) expresses the unitary inequivalence in the infinite volume limit of the flavor and the mass representations and shows the non-trivial nature of the mixing transformations (1), resulting in the condensate structure of the flavor vacuum.
The flavor annihilation operators are defined as $\alpha^r_{\mathbf{k},\sigma}(t) \equiv G_\theta^{-1}(t)\alpha^r_{\mathbf{k},i}G_\theta(t)$ and $\beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \equiv G_\theta^{-1}(t)\beta^{r\dagger}_{-\mathbf{k},i}G_\theta(t)$. For $\mathbf{k} = (0, 0, |\mathbf{k}|)$, we have :

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\, \alpha^r_{\mathbf{k},1} + \sin\theta\, \left(U^*_\mathbf{k}(t)\, \alpha^r_{\mathbf{k},2} + \epsilon^r\, V_\mathbf{k}(t)\, \beta^{r\dagger}_{-\mathbf{k},2} \right) \tag{7}$$
$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\, \alpha^r_{\mathbf{k},2} - \sin\theta\, \left(U_\mathbf{k}(t)\, \alpha^r_{\mathbf{k},1} - \epsilon^r\, V_\mathbf{k}(t)\, \beta^{r\dagger}_{-\mathbf{k},1} \right) \tag{8}$$
$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\, \beta^r_{-\mathbf{k},1} + \sin\theta\, \left(U^*_\mathbf{k}(t)\, \beta^r_{-\mathbf{k},2} - \epsilon^r\, V_\mathbf{k}(t)\, \alpha^{r\dagger}_{\mathbf{k},2} \right) \tag{9}$$
$$\beta^r_{-\mathbf{k},\mu}(t) = \cos\theta\, \beta^r_{-\mathbf{k},2} - \sin\theta\, \left(U_\mathbf{k}(t)\, \beta^r_{-\mathbf{k},1} + \epsilon^r\, V_\mathbf{k}(t)\, \alpha^{r\dagger}_{\mathbf{k},1} \right) \tag{10}$$

where $\epsilon^r = (-1)^r$ and $|U_\mathbf{k}|^2, |V_\mathbf{k}|$ are defined as $U_\mathbf{k}(t) \equiv u^{r\dagger}_{\mathbf{k},2}(t)u^r_{\mathbf{k},1}(t)$, $V_\mathbf{k}(t) \equiv \epsilon^r\, u^{r\dagger}_{\mathbf{k},1}(t)v^r_{-\mathbf{k},2}(t)$, so that

$$|U_\mathbf{k}| = \frac{|\mathbf{k}|^2 + (\omega_{k,1}+m_1)(\omega_{k,2}+m_2)}{2\sqrt{\omega_{k,1}\omega_{k,2}(\omega_{k,1}+m_1)(\omega_{k,2}+m_2)}}, \qquad |U_\mathbf{k}|^2 + |V_\mathbf{k}|^2 = 1. \tag{11}$$

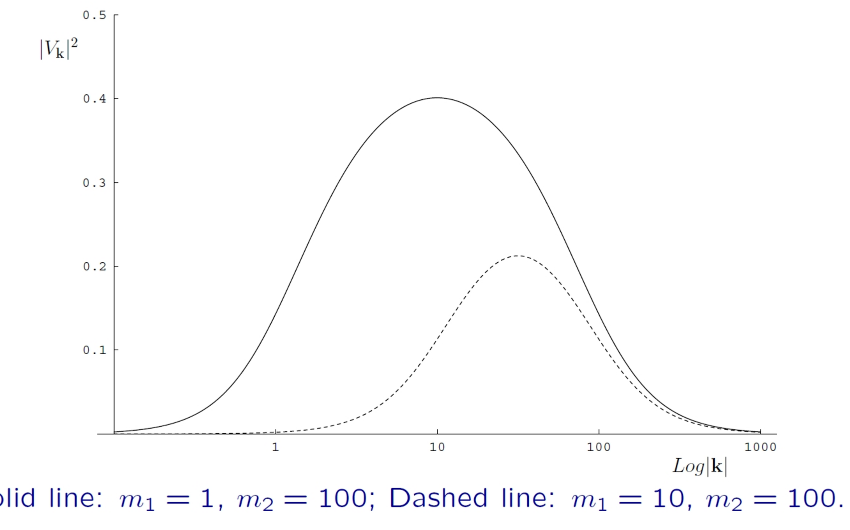
The form of the flavor vacuum is the following one:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k},r} \Big[(1 - \sin^2\theta\, |V_\mathbf{k}|^2) - \epsilon^r \sin\theta\, \cos\theta\, |V_\mathbf{k}|\, (\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}) +$$
$$+ \epsilon^r \sin^2\theta\, |V_\mathbf{k}||U_\mathbf{k}|\, (\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\, |V_\mathbf{k}|^2\, \alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1} \Big]\, |0\rangle_{1,2} \tag{12}$$

The condensation density of the flavor vacuum is given by

$${}_{e,\mu}\langle 0(t)|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^r_{\mathbf{k},i}|0(t)\rangle_{e,\mu} = {}_{e,\mu}\langle 0(t)|\beta^{r\dagger}_{\mathbf{k},i}\beta^r_{\mathbf{k},i}|0(t)\rangle_{e,\mu} = \sin^2\theta\, |V_\mathbf{k}|^2, \qquad i = 1,2\,, \tag{13}$$

Note that the $|V_\mathbf{k}|^2$ has a maximum at $\sqrt{m_1 m_2}$ and $|V_\mathbf{k}|^2 \simeq \frac{(m_2-m_1)^2}{4|\mathbf{k}|^2}$ for $|\mathbf{k}| \gg \sqrt{m_1 m_2}$.



Solid line: $m_1 = 1$, $m_2 = 100$; Dashed line: $m_1 = 10$, $m_2 = 100$.

Disentangling the Mixing Generator

Mixing generator function of m_1, m_2, and θ. Our aim is to disentangle the mass dependence from the one by the mixing angle.

Let us define:

$$R(\theta) \equiv \exp\left\{ \theta \sum_{\mathbf{k},r} \left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^r_{\mathbf{k},2} + \beta^{r\dagger}_{-\mathbf{k},1}\beta^r_{-\mathbf{k},2} \right) e^{i\psi_k} - \left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^r_{\mathbf{k},1} + \beta^{r\dagger}_{-\mathbf{k},2}\beta^r_{-\mathbf{k},1} \right) e^{-i\psi_k} \right] \right\}, \tag{14}$$

$$B_i(\Theta_i) \equiv \exp\left\{ \sum_{\mathbf{k},r} \Theta_{\mathbf{k},i}\, \epsilon^r \left[\alpha^r_{\mathbf{k},i}\beta^r_{-\mathbf{k},i}e^{-i\phi_{k,i}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}e^{i\phi_{k,i}} \right] \right\}, \qquad i = 1,2$$

Since $[B_1, B_2] = 0$ we put

$$B(\Theta_1, \Theta_2) \equiv B_1(\Theta_1)\, B_2(\Theta_2) \tag{15}$$

The $B_i(\Theta_{\mathbf{k},i})$, $i = 1,2$ are ordinary Bogoliubov transformations which introduce a mass shift, and $R(\theta)$ is a rotation.
Their action on the vacuum is given by:

$$|\widetilde{0}\rangle_{1,2} \equiv B^{-1}(\Theta_1,\Theta_2)|0\rangle_{1,2} = \prod_{\mathbf{k},r} \left[\cos\Theta_{\mathbf{k},i} + \epsilon^r \sin\Theta_{\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}\beta^{r\dagger}_{-\mathbf{k},i} \right] |0\rangle_{1,2}$$
$$R^{-1}(\theta)|0\rangle_{1,2} = |0\rangle_{1,2}\,.$$

We find:

$$G_\theta = B(\Theta_1,\Theta_2)\;\; R(\theta)\;\; B^{-1}(\Theta_1,\Theta_2) \tag{16}$$

which is realized when the $\Theta_{\mathbf{k},i}$ are chosen as:

$$\begin{aligned} U_\mathbf{k} &= e^{-i\psi_k}\, \cos(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2}) \\ V_\mathbf{k} &= e^{\frac{(\phi_{k,1}+\phi_{k,2})}{2}}\, \sin(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2}) \end{aligned} \tag{17}$$

and

$$\begin{aligned} \psi_k &= (\omega_{k,1} - \omega_{k,2})\, t \\ \phi_{k,i} &= 2\,\omega_{k,i}\, t \\ \Theta_{\mathbf{k},i} &= \frac{1}{2}\cot^{-1}\left(\frac{|\mathbf{k}|}{m_i} \right) \end{aligned}$$

Then we rewrite the generator as

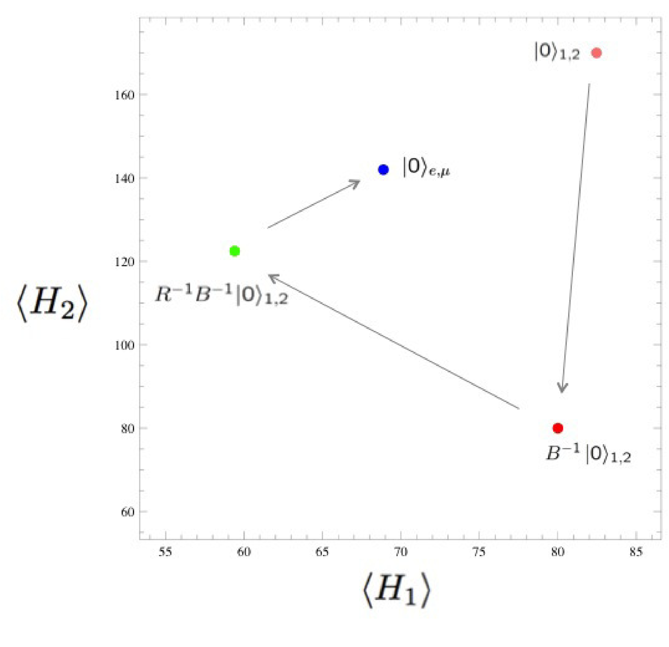
$$G_t(\theta, m_1, m_2) = B_t^{-1}(m_1, m_2)\, R_t(\theta)\, B_t(m_1, m_2) \tag{18}$$

Moreover,

$$|0\rangle_{e,\mu} \equiv G^{-1}|0\rangle_{1,2} = |0\rangle_{1,2} + \Big[B(m_1,m_2)\,,\; R^{-1}(\theta) \Big]\, |\widetilde{0}\rangle_{1,2} \tag{19}$$

which shows that the condensate structure of the flavor vacuum arises as a consequence of the non vanishing commutator $[B, R^{-1}]$.

$\langle H_{\mathbf{k},1} + H_{\mathbf{k},2}\rangle$	State
$-(\omega_{k,1} + \omega_{k,2})$	$\lvert 0\rangle_1 \otimes \lvert 0\rangle_2 \equiv \lvert 0\rangle_{1,2}$
$-(k+k)$	$B^{-1}(m_1,m_2)\lvert 0\rangle_{1,2} \equiv \lvert\widetilde{0}\rangle_{1,2}$
$-k(2\cos^2\theta + (\frac{\omega_{k,1}}{\omega_{k,2}} + \frac{\omega_{k,2}}{\omega_{k,1}})\sin^2\theta)$	$R^{-1}(\theta)B^{-1}(m_1,m_2)\lvert 0\rangle_{1,2} = R^{-1}(\theta)\lvert\widetilde{0}\rangle_{1,2}$
$-(\omega_{k,1} + \omega_{k,2})\big(1 - 2\sin^2\theta\sin^2(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})\big)$	$B(m_1,m_2)R^{-1}(\theta)B^{-1}(m_1,m_2)\lvert 0\rangle_{1,2} \equiv \lvert 0\rangle_{e,\mu}$



Plot of vacuum energies for H_1 and H_2 for the different vacua given in Table for $\theta = \pi/6$, $m_1 = 20$, $m_2 = 150$, $k = 80$.

Conclusions and Outlook

We have shown that it is, indeed, possible to disentangle the dependence of the mixing generator on the mixing angle from the one on the neutrino masses. In fact, it's possible to decompose the generator of flavor mixing transformations for two Dirac fields in terms of two Bogolyubov transformations, depending only by the masses m_1 and m_2, and a rotation, depending only by the mixing angle θ, This result allows for a better understanding of the mixing phenomenon and the associated inequivalence among the mass and flavor representations (Hilbert spaces), in terms of the non-commutativity of the two transformations (Bogoliubov and rotation). This result is interesting because is a little step forward the understanding of the mixing and its origin, in the line of developments achieved in Ref. [3] .

References

[1] M. Blasone and G. Vitiello, Annals Phys. **244** (1995) 283
[2] A. Perelomov, *Generalized Coherent States and Their Applications* - (Springer–Verlag, Berlin, 1986).
[3] M. Blasone, P. Jizba, G. Lambiase and N. E. Mavromatos, arXiv:1312.4924 [hep-ph].

[1]For simplicity, we limit ourselves to the case of two generations (flavors) although the main results presented below have general validity.
[2]We refer to neutrinos, but the discussion is clearly valid for any Dirac fields.
[3]$\{\nu_i^\alpha(x), \nu_j^{\beta\dagger}(y)\}_{t=t'} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}\delta_{ij}\,, \quad \alpha,\beta = 1,..,4\,, \quad \{\alpha^r_{\mathbf{k},i}, \alpha^{s\dagger}_{\mathbf{q},j}\} = \delta_{\mathbf{k}q}\delta_{rs}\delta_{ij}; \quad \{\beta^r_{\mathbf{k},i}, \beta^{s\dagger}_{\mathbf{q},j}\} = \delta_{\mathbf{k}q}\delta_{rs}\delta_{ij}, \quad i,j = 1,2\,.$