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Mapping the geometry of exceptional Lie groups as symmetries of physical systems

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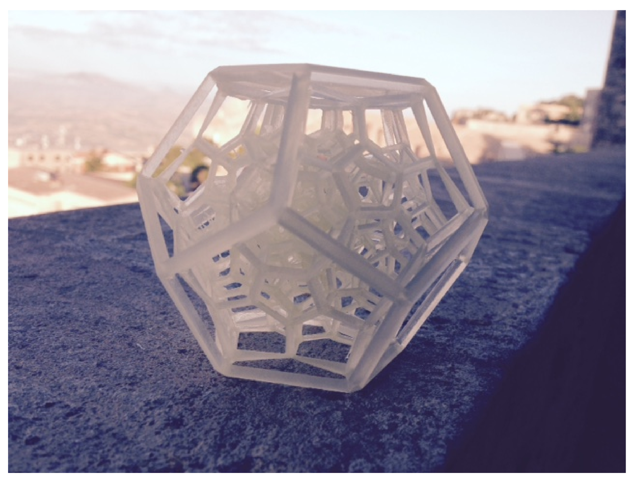
http://www.mat.unimi.it/users/cerchiai/MathematicaProgram/,
preprint LMU-ASC 78/14, based also on previous works with
S. Cacciatori, S. Ferrara, A. Marrani, arXiv:1402.5063;
S. Cacciatori, A. Marrani, arXiv:1208.6153, 1202.3055, 1201.6314, 1201.6667;
B. Van Geemen, arXiv: 1003.4255; S. Cacciatori, arXiv: 0906.0121;
S. Cacciatori, A. Scotti et al., arXiv:0710.0356, 0705.3978, hep-th/0503106



Abstract

For the last several years I have been pursuing a project of mapping the geometry of the exceptional Lie groups $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ and how they act as symmetries of different physical models. The method used to construct the corresponding Lie algebras utilizes the Tits formula, which exploits the Jordan algebraic structure, arranging the groups into magic squares and highlighting the role of the octonions. I have written a Mathematica program to generate the fundamental smallest-dimensional irreducible representations and the adjoint of each of the exceptional Lie algebras. I have also developed an algebraic method to determine the global structure of the full group, which for the compact form is a generalization of the Euler angles for $SU(2)$, while for the non compact forms it is based on the Iwasawa decomposition.

Then I have applied this knowledge of the geometric structure of the exceptional Lie groups to the investigation of their action on various physical models, such as supergravity, where they describe the electric magnetic duality of the theory as well as the stabilizer of the scalar manifolds and of certain black hole orbits under the attractor mechanism. I have analyzed the structure of the embeddings of certain $\mathfrak{sl}(2)$ subgroups, such as the principal triple, which determines the cohomology of the group. Such $\mathfrak{sl}(2)$ subgroups are relevant for quantum information theory and for confinement in the corresponding gauge theories.

Plan

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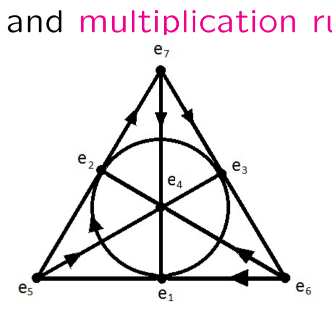
1. Exceptional Lie groups and normed division algebras
(a) Normed division algebras and split composition algebras

The octonions can be defined by: $\mathbb{O} \cong \langle 1, e_1, \ldots, e_7\rangle_{\mathbb{R}}$,
with e_i imaginary units: $e_i^2 = -1$, 1 the identity: $1\, e_i = e_i 1 = e_i$
and multiplication rule described by the Fano plane:



Let (e_i, e_j, e_k) be an ordered triple of points on a given line in the order of the arrow. Then: $e_i e_j = e_k, \qquad e_j e_i = -e_k$.

The split octonions $\mathbb{O}_S$ can be obtained e.g. by substituting $e_i \to \tilde{e}_i$, $i = 4,5,6,7$, so that $\tilde{e}_i^2 = 1$ instead of $e_i^2 = -1$:

$$\mathbb{O}_S \cong \langle 1, e_1, e_2, e_3, \tilde{e}_4, \ldots, \tilde{e}_7\rangle_{\mathbb{R}}.$$

Subalgebras corresponding to the quaternions $\mathbb{H}$ and the complex numbers $\mathbb{C}$ are e.g.: $\mathbb{H} \cong \langle 1, e_1, e_5, e_6\rangle_{\mathbb{R}}$ and $\mathbb{C} \cong \langle 1, e_5\rangle_{\mathbb{R}}$.
The split quaternions $\mathbb{H}_S$ and the split complex numbers $\mathbb{C}_S$ can be constructed as: $\mathbb{H}_S \cong \langle 1, e_1, \tilde{e}_5, \tilde{e}_6\rangle_{\mathbb{R}}$ and $\mathbb{C}_S \cong \langle 1, \tilde{e}_5\rangle_{\mathbb{R}}$.

(b) Vinberg's formula

Vinberg's formula [Onishchik, Vinberg, Lie Groups and Lie Algebras III, Springer, Berlin, 1991] associates to the division algebras $\mathbb{A}$ and $\mathbb{B}$ the Lie algebra $\mathcal{L}(\mathbb{A},\mathbb{B})$ defined as:

$$\mathcal{L}(\mathbb{A},\mathbb{B}) = \mathrm{tri}(\mathbb{A}) \oplus \mathrm{tri}(\mathbb{B}) \dotplus 3\mathbb{A}\otimes\mathbb{B},$$

where $\mathrm{Tri}(\mathbb{A})$ is the triality, generalizing the concept of derivations:

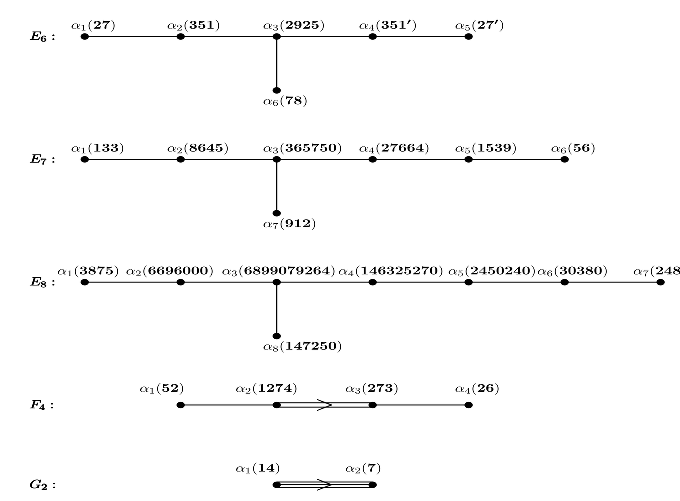
$$\mathrm{Tri}(\mathbb{A}) = \{(A,B,C) \text{ with } A,B,C \in \mathrm{End}(\mathbb{A}) \quad | \quad A(x_1x_2) = B(x_1)x_2 + x_1C(x_2)\}.$$

Vinberg's formula is manifestly symmetric: $\mathcal{L}(\mathbb{A},\mathbb{B}) = \mathcal{L}(\mathbb{B},\mathbb{A})$.
It can be rewritten in terms of symmetries of Jordan algebras, yielding the equivalent Tits' formula:

$$\mathcal{L}(\mathbb{A},\mathbb{B}) = \mathrm{Der}(\mathbb{A}) \oplus \mathrm{Der}(\mathfrak{J}(\mathbb{B})) \oplus \left(\mathbb{A}^0 \otimes \mathfrak{J}^0(\mathbb{B})\right).$$

Here $\mathrm{Der}(\mathbb{A})$ is the algebra of the derivations of $\mathbb{A}$;
$\mathfrak{J}(\mathbb{B})$ is the Jordan algebra on $\mathbb{B}$, i.e. the set of all 3×3 matrices J with entries in $\mathbb{B}$ satisfying $\eta J^\dagger \eta = J$, where η is the metric (Euclidean or Lorentzian) of the Jordan algebra $\mathfrak{J}(\mathbb{B})$, and the Jordan product $\circ$ is defined as the symmetrized matrix multiplication;
$\mathbb{A}^0$ is the set of the traceless elements of $\mathbb{A}$.

(c) Dynkin Diagrams and Fundamental Representations



Lie groups are classified by the Dynkin diagram, describing the structure of their root system and determining the fundamental representations. All of the forms of the exceptional Lie groups can be recovered by means of the Tits' formula. I have written a Mathematica program to construct the fundamental smallest-dimensional irreducible representations and the adjoint.

(d) Magic squares

Vinberg's formula allows to arrange the Lie algebras it defines into magic squares.

The Freudenthal-Tits magic compact square $\mathcal{L}_3(\mathbb{A},\mathbb{B})$

[Freudenthal, Adv. Math. **1**, 145 (1963); Tits, Nederl. Akad. Wetensch. Proc. Ser. **A 69**, 223 (1966)]

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	SO(3)	$SU(3)$	$USp(6)$	$F_{4(-52)}$
$\mathbb{C}$	$SU(3)$	$SU(3)\times SU(3)$	$SU(6)$	$E_{6(-78)}$
$\mathbb{H}$	$USp(6)$	$SU(6)$	$SO(12)$	$E_{7(-133)}$
$\mathbb{O}$	$F_{4(-52)}$	$E_{6(-78)}$	$E_{7(-133)}$	$E_{8(-248)}$

The single split Günaydin-Sierra-Townsend magic square $\mathcal{L}_3(\mathbb{A}_S,\mathbb{B})$

[Günaydin, Sierra, Townsend, Phys. Lett. **133B** (1983) 72]

			$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathrm{Aut}(\mathfrak{J}_3(\mathbb{B}))$	$\to$	$\mathbb{R}$	$SO(3)$	$SU(3)$	$USp(6)$	$F_{4(-52)}$
$\mathrm{Str}_0(\mathfrak{J}_3(\mathbb{B}))$	$\to$	$\mathbb{C}_S$	$SL(3,\mathbb{R})$	$SL(3,\mathbb{C})$	$SU^*(6)$	$E_{6(-26)}$
$\mathrm{Conf}(\mathfrak{J}_3(\mathbb{B}))$	$\to$	$\mathbb{H}_S$	$Sp(6,\mathbb{R})$	$SU(3,3)$	$SO^*(12)$	$E_{7(-25)}$
$\mathrm{QConf}(\mathfrak{J}_3(\mathbb{B}))$	$\to$	$\mathbb{O}_S$	$F_{4(4)}$	$E_{6(2)}$	$E_{7(-5)}$	$E_{8(-24)}$

The double split Barton-Sudbery magic square $\mathcal{L}_3(\mathbb{A}_S,\mathbb{B}_S)$

[Barton, Sudbery, Adv. in Math. **180**, 596 (2003), math/0203010 [math.RA]]

	$\mathbb{R}$	$\mathbb{C}_S$	$\mathbb{H}_S$	$\mathbb{O}_S$
$\mathbb{R}$	SO(3)	$SL(3,\mathbb{R})$	$Sp(6,\mathbb{R})$	$F_{4(4)}$
$\mathbb{C}_S$	$SL(3,\mathbb{R})$	$SL(3,\mathbb{R})\times SL(3,\mathbb{R})$	$SL(6,\mathbb{R})$	$E_{6(6)}$
$\mathbb{H}_S$	$Sp(6,\mathbb{R})$	$SL(6,\mathbb{R})$	$SO(6,6)$	$E_{7(7)}$
$\mathbb{O}_S$	$F_{4(4)}$	$E_{6(6)}$	$E_{7(7)}$	$E_{8(8)}$

By means of Lorentzian Jordan algebras and the corresponding magic square, I have been able to recover also $F_{4(-20)}$ and $E_{6(-14)}$:

The Lorentzian non split magic square $\mathcal{L}_{(1,2)}(\mathbb{A},\mathbb{B})$

[with S. Cacciatori and A. Marrani, arXiv:1208.6153 [math-ph]]

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$SL(2,\mathbb{R})$	$SU(2,1)$	$USp(4,2)$	$F_{4(-20)}$
$\mathbb{C}$	$SU(2,1)$	$SU(2,1)\times SU(2,1)$	$SU(4,2)$	$E_{6(-14)}$
$\mathbb{H}$	$USp(4,2)$	$SU(4,2)$	$SO(8,4)$	$E_{7(-5)}$
$\mathbb{O}$	$F_{4(-20)}$	$E_{6(-14)}$	$E_{7(-5)}$	$E_{8(8)}$

2. Global parametrizations of Lie groups: The Iwasawa parametrization for the noncompact form and the Euler angles for the compact form [with S. Cacciatori, arXiv:0906.0121 [math-ph]]

1) Identification of the Lie algebra $\mathfrak{h}$ corresponding, respectively, to the symmetrically embedded maximal compact subgroup H for the Iwasawa parametrization, or the symmetrically embedded maximal subgroup H with respect to which the Euler angles are constructed:
$\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{p}$, with $\mathfrak{p}$ the vector space orthogonal to $\mathfrak{h}$ and $h = \dim(H)$.
2) Choice of a Cartan subalgebra $\mathfrak{a}$ as a pivot, in such a way as to have the maximal number of generators in G/H. It is generated by l commuting generators $\{c_1,\ldots,c_l\}$, where l is the rank of the group. Out of these, at most r can be chosen outside of H, i.e. $\{c_1,\ldots,c_r\}$, where r is the rank of the coset G/H:
$\mathfrak{a} = \mathfrak{a}_{\mathfrak{h}}\oplus\mathfrak{a}_{\mathfrak{p}}$, with $\mathfrak{a}_{\mathfrak{p}} = \mathfrak{a}\cap\mathfrak{p}$, $\mathfrak{a}_{\mathfrak{h}}\subset\mathfrak{h}$, $\dim(\mathfrak{a}_{\mathfrak{p}}) = r$, $\dim(\mathfrak{a}_{\mathfrak{h}}) = s = l - r$.
3) Calculation of the corresponding system of positive roots $\{\alpha_i\}$ and of the corresponding eigenmatrices $\{\lambda_{\alpha_i}\}$, $i = 1,\ldots h-k$, with k the dimension of the normalizer $\mathfrak{k}$ of $\mathfrak{a}_{\mathfrak{p}}$ in $\mathfrak{h}$.
4) Computation of a realization of a generic element g of the group as:
$$g = HAB$$
with $H =$ the subgroup corresponding to the fiber,
$A =$ the Abelian subgroup generated by $\mathfrak{a}_p$,
$$B = \begin{cases} N \text{ nilpotent subgroup generated by } \{\lambda_{\alpha_i}\} & \text{for the Iwasawa param.;}\\ H/K \text{ with } K \text{ the subgroup generated by } \mathfrak{k} & \text{for the Euler angles.}\end{cases}$$
5) For the Euler angles, choice of the range of the coordinates y_i, $i = 1,\ldots,r$ corresponding to the generators of A: $0 \le \vec{\alpha}_i\cdot\vec{y} \le \pi$, $\quad 0 \le \sum_{i=1}^r n_i\vec{\alpha}_i\cdot\vec{y} \le \pi$ with $\sum_{i=1}^r n_i\vec{\alpha}_i$ the highest root of G/H.

3. Applications to supergravity

Exceptional Lie groups are relevant for extended supergravity theories, where they enter as the electric-magnetic dualities or the stabilizers/isotropy groups of scalar manifolds and of certain black hole orbits under the attractor mechanism. (For a review of symmetric spaces in supergravity see e.g. [Ferrara, Marrani, arXiv:0808.3567] or [Boya, arXiv:0811.0554]). Magic squares describe part of the web of dimensional reductions and truncations of supergravities in various space-time dimensions.

Examples studied in the framework of this program:

(a) Lorentzian Jordan algebras and the corresponding magic squares $\Longrightarrow$ Construction of $F_{4(-20)}$ and of $E_{6(-14)}$, relevant for supergravity, e.g. the latter being the stabilizer of the large non-BPS U-orbit with vanishing central charge of the magic supergravity in $D = (3,1)$ dimensions.
[with Cacciatori, Marrani, arXiv:1208.6153 [math-ph]]

(b) Classification of charge orbits for black holes in 4-dim. [with Marrani, Ferrara, Zumino, arXiv:0902.3973 [hep-th]] and 5-dim. supergravity [with Marrani, Ferrara, Zumino, arXiv:1006.3101 [hep-th]]

(c) Iwasawa parametrization and E_6-covariant formulation for $\frac{E_{7(-25)}}{E_6\times U(1)/\mathbb{Z}_3}$, relevant for magic $\mathcal{N} = 2$ supergravity in $D = 4$ space-time dimensions $\Longrightarrow$ The E_6-covariant expression is the analogue of the Calabi-Vesentini coordinates, exhibiting the maximal possible covariance, while the Iwasawa decomposition is SO(8)-covariant, highlighting the role of triality.
[with Cacciatori, Marrani, arXiv:1201.6314 [hep-th]]

(d) Iwasawa parametrization of $E_{7(7)}$

[with S.L. Cacciatori, A. Marrani, arXiv:1005.2231, 1202.3055 [hep-th]]

- It is relevant for $\mathcal{N} = 8$ supergravity in $D = 4$ space-time dimensions [Cremmer, Julia, Nucl. Phys. **B159**, 141 (1979)].
- It is one of the four real forms of E_7: it is the split form (i.e. maximally non compact form). The second subscript in the name $E_{7(7)}$ indicates the difference between the number of non compact (70) and of compact (63) generators. Its maximal compact subgroup is given by $\frac{SU(8)}{\mathbb{Z}_2}$.
- By exploiting its symplectic structure, following [Adams, *"Lectures on exceptional Lie groups"*, The University of Chicago Press, Chicago and London (1996)], its fundamental representation can be constructed as the algebra $\mathfrak{sl}(V)\oplus\Lambda^4V^4$ acting on $\Lambda^2V\oplus\Lambda^2V^*$ with $V = \mathbb{R}^8$, V^* its dual and $\wedge$ the wedge product.
- Define the 56 x 56 matrices ($1 \le i < j \le 8, 1 \le k < l \le 8, 1 \le m < n \le 8$):
$M_{kl}^{\pm} = \sum_{m,n}(U_{klim}^{\pm}D_{kljn} + D_{klim}U_{kljn}^{\pm})$ with
$U_{klim}^{\pm} \equiv \delta_{km}\delta_{li} \pm \delta_{ki}\delta_{lm}$ and $D_{klim} \equiv \begin{cases}\delta_{im} \text{ for } k\neq l\neq i;\\ 0 \text{ otherwise.}\end{cases}$,
$\mathcal{M}_I^{\pm} \equiv \frac{1}{\sqrt{2}}(\lambda_I \pm \lambda_I^T)$, with $\lambda_{i_1i_2i_3i_4} = \begin{pmatrix} 0 & \epsilon_{i_1i_2i_3i_4ijkl}\\ \delta^{ijkl}_{i_1i_2i_3i_4} & 0\end{pmatrix}$, and
$h_{D_\alpha}, (\alpha = 1,\ldots,7)$ generated by the embedding of the 7 diagonal traceless 8 x 8 matrices D_α in $\mathbb{R}^{56}\times\mathbb{R}^{56}$.
Then $SU(8)$ is generated by the 63 antisymmetric matrices $\{M_{kl}^-, \mathcal{M}_{I_8}^-\}$, while $\frac{E_{7(7)}}{SU(8)/\mathbb{Z}_2}$ is generated by the 70 symmetric matrices $\{M_{kl}^+, \mathcal{M}_{I_8}^+, D_\alpha\}$, where $I_8 = \{i_1i_2i_38\}$.

4. Application to quantum information theory and the embedding of $\mathfrak{sl}(2)^7$ in $\mathfrak{e}_7$

[with B. Van Geemen, arXiv:1003.4255 [quant-ph]]

Motivation: Relation between quantum information theory and supergravity discovered by Duff and Ferrara [Duff, Ferrara, quant-ph/0609227, hep-th/0612036, arXiv:0704.0507], linking entanglement measures for qubits to black hole entropy.

Qubits in quantum information theory

k-qubits: non-zero elements of the finite abelian group $L_k = \mathbb{Z}_k$

State space $\mathcal{H}_k$: 2^k-dim. vector space of $\mathbb{C}$-valued maps $L_k \to \mathbb{C}$

Action of the qubits on this state space can be extended to an action of the group generated by the generalized Pauli matrices, denoted as the Heisenberg group H_k, whose normalizer coincides with a quotient of the Weyl group of E_7 by $\mathbb{Z}_2$.

$\Longrightarrow$ Study of emebddings of $\mathfrak{sl}(2)^7$ in $\mathfrak{e}_7$ as Lagrangian subspaces of $\mathbb{Z}_2^6$, on which the Weyl group is acting,

$\Longrightarrow$ Proof that there are 135 such embeddings and computation of their weights.

5. Study of $\mathfrak{sl}(2)$ subgroups

Following Dynkin [Dynkin, Mat. Sb. (N.S.), **30(72)**:349–462 (1952)] I have computed [with Cacciatori,Ferrara,Marrani, arXiv:1402.5063 [hep-th]] the principal [B. Kostant, Am. J. Math. **81**, 973–1032 (1959)] and then the maximal subgroups of type A_1, i.e. $SL(2)$ or $SU(2)$, for all the exceptional Lie groups.

Method: Given a Lie algebra $\mathfrak{g}$ of rank l with Cartan subalgebra $\mathfrak{a}^{\mathfrak{g}}$, the Cartan subalgebra $\mathfrak{a}^{\mathfrak{k}}$ of a subgroup $\mathfrak{k}$ of rank r can be chosen as: $\mathfrak{a}^{\mathfrak{k}} = \mathfrak{P}\mathfrak{a}^{\mathfrak{g}}$, with $\mathfrak{P}$ called the projection matrix. For a $\mathfrak{sl}(2)$ subgroup of rank 1, it can be found by solving the linear system: $(h,\alpha_i) = d_i, i = 1,\ldots,l$, with h its Cartan generator and $\vec{d} = (d_1,\ldots,d_l)$ its Dynkin vector.

Using my Mathematica package, I have obtained explicit expressions for the generators of the above A_1 subgroups and implemented the calculation of their branching rules.

G	Dynkin Vector	Projection Vector	Branching of smallest fundamental	Branching of adjoint
E_7	2 2 2 0 2 2 2	21 40 57 72 50 26 37	$\frac{21}{2}$ $\frac{15}{2}$ $\frac{11}{2}$ $\frac{5}{2}$	13 11 9 8 7 2×5 3 1
	2 2 2 2 2 2 2	27 52 75 96 66 34 49	$\frac{27}{2}$ $\frac{17}{2}$ $\frac{9}{2}$	17 13 11 9 7 5 1
	2 2 0 2 0 2 2 2	38 74 108 142 174 118 60 88	19 17 14 13 2×11 9 8 7 5 3 1	
E_8	2 2 2 2 0 2 2 2	46 90 132 172 210 142 72 106	23 19 17 14 13 11 9 7 5 1	
	2 2 2 2 2 2 2 2	58 114 168 220 270 182 92 136	29 23 19 17 13 11 7 1	

Conclusions and outlook

- In a project to map the geometry of the exceptional Lie groups, I have developed a Wolfram Mathematica program (http://www.mat.unimi.it/users/cerchiai/MathematicaProgram/, LMU-ASC 78/14, work in progress), which generates the fundamental smallest-dim. irreducible representations and the adjoint of all the exceptional Lie algebras, as well as the Cartan subalgebras and their root systems.
- It has been applied e.g. to the study of some novel Lorentzian magic squares, which allow to explicitly construct also $F_{4(-20)}$ and $E_{6(-14)}$ and appears in the study of supergravity of square of Super-Yang-Mills in 3 dimensions [Borsten, Duff, Hughes, Nagy, arXiv:1301.4176 [hep-th]].
- Starting from the explicit expressions for Lie algebras of the exceptional Lie groups, global parametrizations of the cosets $\frac{E_{7(-25)}}{E_6\times U(1)/\mathbb{Z}_3}$, relevant for magic $\mathcal{N} = 2$ supergravity in $D = 4$, and $\frac{E_{7(7)}}{SU(8)/\mathbb{Z}_2}$ relevant for $\mathcal{N} = 8$ supergravity in $D = 4$, have been constructed.
- The explicit computation of subgroups is possible, e.g. of $\mathfrak{sl}(2)$ subalgebras. These generate nilpotent orbits, relevant e.g. for the study calorons (see e.g. [Diakonov, Petrov, arXiv:1011.5636 [hep-th]]) in the framework of confinement [Cossu, D'Elia, Di Giacomo, Lucini, Pica, arXiv:0710.0481 [hep-lat]].
- Other embeddings that can be studied are e.g. $\mathfrak{sl}(2)^7$ in $\mathfrak{e}_7$, which is important for quantum information theory as well as defining a "curious" 4-dim $\mathcal{N} = 1$ supergravity [Duff, Ferrara, arXiv:1010.3173 [hep-th]] with coset $SL(2)^7/SO(2)^7$.
- Construction of non symmetric embeddings, such as e.g. $\frac{\mathfrak{g}_{2(2)}}{\mathfrak{sl}(3)}$ or $\frac{\mathfrak{f}_{4(4)}}{\mathfrak{sl}(3)\times\mathfrak{sl}(3)}$ is feasible. This is relevant for the determination of their properties in geometry and for supergravity in physics.