Heavy flavor contributions to deep-inelastic scattering at 3-loop order

A. Behring

DESY

June 26th, 2015

based on
[J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. Hasselhuhn, A. von Manteuffel, M. Round, C. Schneider, F. Wißbrock '14 [Nucl. Phys. B 886 (2014) 733]],

[J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, C. Schneider '14 [Nucl. Phys. B 890 (2014) 48]] and

[A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel, C. Schneider '15 [Nucl.Phys. B 897 (2015) 612]]

Motivation

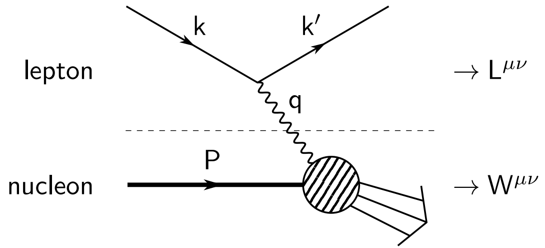
Experiments at hadron colliders like LHC

- Predictions depend on fundamental parameters to be determined from experiment
  $\Rightarrow$ e.g. Higgs production is very sensitive to gluon PDF and α_s
- Some of these parameters, like α_s, m_c and PDFs can be extracted from deep-inelastic scattering data

Deep-inelastic scattering as a tool for LHC

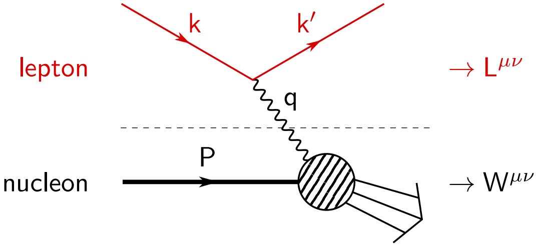
- Current accuracy of world data requires next-to-next-to-leading order (NNLO) analysis
- Heavy flavor (e.g. charm) contributions have different scale evolution than massless partons $\Rightarrow$ handle on gluon distribution
- Heavy flavor contributions are not yet known at NNLO
  $\Rightarrow$ Their calculation will be the topic of this talk

Heavy flavor contributions to deep-inelastic scattering



Cross section: $$\frac{\mathrm{d}\sigma}{\mathrm{d}x\mathrm{d}Q^2} \propto L_{\mu\nu}\, W^{\mu\nu}$$

Heavy flavor contributions to deep-inelastic scattering



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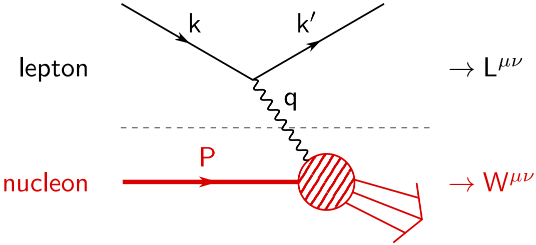
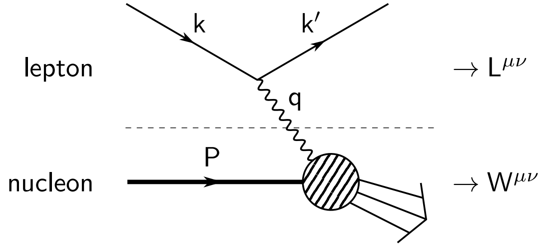
Heavy flavor contributions to deep-inelastic scattering



Cross section: $$\frac{d\sigma}{dx\,dQ^2} \propto L_{\mu\nu} W^{\mu\nu}$$

Hadronic tensor: $$W_{\mu\nu} = (\dots)_{\mu\nu} F_L(x, Q^2) + (\dots)_{\mu\nu} F_2(x, Q^2)$$

Heavy flavor contributions to deep-inelastic scattering



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Structure functions contain light and heavy quark contributions.

Heavy flavor contributions to deep-inelastic scattering

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Structure functions: $F_2(x) = x \sum_j \mathbb{C}_{2,j}(x) \otimes f_j(x)$

Wilson coefficients
perturbative

PDFs
non-perturbative

Heavy flavor contributions to deep-inelastic scattering

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x- and N-space are connected by a Mellin transformation:

$$M[f(x)](N) = \int_0^1 \mathrm{d}x\, x^{N-1} f(x)$$

Representation simplifies in Mellin space.

Heavy flavor contributions to deep-inelastic scattering

Hadronic tensor: $W_{\mu\nu} = (\dots)_{\mu\nu} F_L(x, Q^2) + (\dots)_{\mu\nu} F_2(x, Q^2)$

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Wilson coefficients: $\boxed{\mathbb{C}_{2,j}(N) = C_{2,j}(N) + H_{2,j}(N)}$

massless Wilson coefficients ($C_{2,j}$)
NNLO: [Moch, Vermaseren, Vogt '05]

heavy-flavor Wilson coefficients ($H_{2,j}$)

Heavy flavor contributions to deep-inelastic scattering

Hadronic tensor: $W_{\mu\nu} = (\dots)_{\mu\nu} F_L(x, Q^2) + (\dots)_{\mu\nu} F_2(x, Q^2)$

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Wilson coefficients: $\mathbb{C}_{2,j}(N) = C_{2,j}(N) + \boxed{H_{2,j}(N)}$

For $Q^2/m^2 \geq 10$ the heavy flavor Wilson coefficients factorize:
[Buza, Matiounine, Smith, Migneron, van Neerven '96]

Heavy flavor Wilson coefficients: $$H_{2,j}(N) = \sum_i A_{ij}(N) C_{2,i}(N)$$

massive operator matrix elements (OMEs) ($A_{ij}(N)$)

massless Wilson coefficients ($C_{2,i}(N)$)

LO: [Witten '76; Babcock, Sievers '78; Shifman, Vainshtein, Zakharov '78; Leveille, Weiler '79; Glück, Reya '79; Glück, Hoffmann, Reya '82]
NLO: [Laenen, van Neerven, Riemersma, Smith '93; Buza, Matiounine, Smith, Migneron, van Neerven '96; Bierenbaum, Blümlein, Klein '07, '08, '09]

Heavy flavor contributions to deep-inelastic scattering

Hadronic tensor: $W_{\mu\nu} = (\dots)_{\mu\nu} \, F_L(x, Q^2) + (\dots)_{\mu\nu} \, F_2(x, Q^2)$

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Heavy flavor Wilson coefficients: $H_{2,j}(N) = \sum_i A_{ij}(N) C_{2,i}(N)$

OMEs A_{ij} also essential to define the variable flavor number scheme
→ describe transition from n_f to $n_f + 1$ massless quarks
→ transitions relevant for the PDFs at the LHC

Status of Wilson coefficients and OMEs

Moments for F_2: $N = 2 \ldots 10(14)$ ✓ [Bierenbaum, Blümlein, Klein, 04/2009]

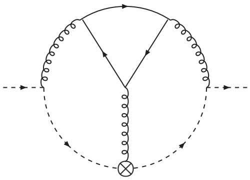
Massive operator matrix elements at NNLO

- A_{Qg} work in progress
- A_{gg} ✓
- A_{Qq}^{PS} ✓ → this talk
- $A_{qq,Q}^{\mathrm{NS}}$ ✓ [Ablinger et al. 06/2014]
- $A_{qq,Q}^{\mathrm{TR}}$ ✓ [Ablinger et al. 06/2014]
- $A_{gq,Q}$ ✓ [Ablinger et al. 02/2014]
- $A_{qg,Q}$ ✓ [Ablinger et al. 08/2010]
- $A_{qq,Q}^{\mathrm{PS}}$ ✓ [Ablinger et al. 08/2010]

Heavy flavor Wilson coefficients at NNLO

- $H_{g,2}^{\mathrm{S}}$ work in progress
- $H_{q,2}^{\mathrm{PS}}$ ✓ → this talk
- $L_{q,2}^{\mathrm{NS}}$ ✓ → this talk
- $L_{q,2}^{\mathrm{PS}}$ ✓ [Behring et al. 03/2014]
- $L_{g,2}^{\mathrm{S}}$ ✓ [Behring et al. 03/2014]

Calculating massive operator matrix elements



Definition of the OMEs A_{ij}

$$A_{ij} := \langle j | O_i | j \rangle$$

O_i: local light-cone operators
$|j\rangle$: partonic states (massless, on-shell)

Example: $O^{\mathrm{NS}}_{q,a;\mu_1,\ldots,\mu_N} = i^{N-1} S[\bar{\Psi}\gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_N} \frac{\lambda^a}{2} \Psi] -$ trace terms

Outline of the computation

- Generate diagrams (`QGRAF`) [P. Nogueira '93]
- Apply Feynman rules including operators; $(\Delta.p)^N \rightarrow \frac{1}{1 - x\Delta.p}$
- Reduce to master integrals (extension of `Reduze 2`) [von Manteuffel, Studerus '10,'12]
- Solve the master integrals ($\rightarrow$ next slide)
- Put everything together and create results in N- and x-space

Solving the master integrals

Solve the master integrals applying

- higher hypergeometric functions,
- Mellin-Barnes integrals,
- Almkvist-Zeilberger algorithm,
- difference equations.

Transform resulting sum representations using

- Sigma [Schneider '05-], HarmonicSums [Ablinger, Blümlein, Schneider '10,'13],
- EvaluateMultiSums & SumProduction [Ablinger, Blümlein, Hasselhuhn, Schneider '10-].

Results can be expressed in terms of nested sums in N-space
→ see next slide

Example for a result: $a_{Qq}^{\mathrm{PS},(3)}$

$$a_{Qq}^{(3),\mathrm{PS}}(N) =$$
$$C_F T_F^2\Bigg[\frac{32}{27(N-1)(N+3)(N+4)(N+5)}\Bigg(\frac{P_{15}}{N^3(N+1)^2(N+2)^2}S_2$$
$$-\frac{P_{19}}{N^3(N+1)^3(N+2)^2}S_1^2+\frac{2P_{28}}{3N^4(N+1)^4(N+2)^3}S_1-\frac{2P_{32}}{9N^5(N+1)^4(N+2)^4}\Bigg)$$
$$-\frac{32P_3}{9(N-1)N^3(N+1)^2(N+2)^2}\zeta_2+\Bigg(\frac{32}{27}S_1^3-\frac{160}{9}S_1S_2-\frac{512}{27}S_3+\frac{128}{3}S_{2,1}$$
$$+\frac{32}{3}S_1\zeta_2-\frac{1024}{9}\zeta_3\Bigg)F\Bigg]+C_F N_F T_F^2\Bigg[\frac{16P_7}{27(N-1)N^3(N+1)^3(N+2)^2}S_1^2$$
$$+\frac{208P_7}{27(N-1)N^3(N+1)^3(N+2)^2}S_2-\frac{32P_{21}}{81(N-1)N^4(N+1)^4(N+2)^3}S_1$$
$$+\frac{32P_{29}}{243(N-1)N^5(N+1)^5(N+2)^4}+\Bigg(-\frac{16}{27}S_1^3-\frac{208}{9}S_1S_2-\frac{1760}{27}S_3-\frac{16}{3}S_1\zeta_2$$
$$+\frac{224}{9}\zeta_3\Bigg)F+\frac{1}{(N-1)N^3(N+1)^3(N+2)^2}\frac{16P_7}{9}\zeta_2\Bigg]$$
$$+C_F^2T_F\Bigg[\frac{32P_9}{3(N-1)N^3(N+1)^3(N+2)^2}S_{2,1}-\frac{16P_{14}}{9(N-1)N^3(N+1)^3(N+2)^2}S_3$$
$$-\frac{4P_{17}}{3(N-1)N^4(N+1)^4(N+2)^3}S_1^2+\frac{4P_{23}}{3(N-1)N^4(N+1)^4(N+2)^3}S_2$$
$$+\frac{4P_{31}}{3(N-1)N^6(N+1)^6(N+2)^4}+\Bigg(\Bigg(\frac{2P_5}{N^2(N+1)^2}-\frac{4P_1}{N(N+1)}S_1\Bigg)\zeta_2$$
$$-\frac{4P_1}{9N(N+1)}S_1^3\Bigg)G+\Bigg(\Bigg(\frac{80}{9}S_3-64S_{2,1}\Bigg)S_1-\frac{2}{9}S_1^4-\frac{20}{3}S_1^2S_2+\frac{46}{3}S_2^2+\frac{124}{3}S_4$$
$$+\frac{416}{3}S_{2,1,1}+64\Bigg(\Bigg(S_3(2)-S_{1,2}(2,1)+S_{2,1}(2,1)-S_{1,1,1}(2,1,1)\Bigg)S_1\left(\frac{1}{2}\right)$$
$$-S_{1,3}\left(2,\frac{1}{2}\right)+S_{2,2}\left(2,\frac{1}{2}\right)-S_{3,1}\left(2,\frac{1}{2}\right)+S_{1,1,2}\left(2,\frac{1}{2},1\right)-S_{1,1,2}\left(2,1,\frac{1}{2}\right)$$
$$-S_{1,2,1}\left(2,\frac{1}{2},1\right)+S_{1,2,1}\left(2,1,\frac{1}{2}\right)-S_{2,1,1}\left(2,\frac{1}{2},1\right)-S_{2,1,1}\left(2,1,\frac{1}{2}\right)$$
$$+S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)+S_{1,1,1,1}\left(2,1,\frac{1}{2},1\right)+S_{1,1,1,1}\left(2,1,1,\frac{1}{2}\right)\Bigg)\Bigg)+\Big(12S_2-4S_1^2\Big)\zeta_2$$
$$+\Bigg(\frac{112}{3}S_1-448S_1\left(\frac{1}{2}\right)\Bigg)\zeta_3+144\zeta_4-32B_4\Bigg)F+\frac{32P_2 2^{-N}}{(N-1)N^3(N+1)^2}\Bigg(-S_3(2)$$
$$+S_{1,2}(2,1)-S_{2,1}(2,1)+S_{1,1,1}(2,1,1)+7\zeta_3\Bigg)+\Bigg(-\frac{4P_8}{3(N-1)N^3(N+1)^3(N+2)^2}S_2$$
$$+\frac{8P_{27}}{3(N-1)N^5(N+1)^5(N+2)^4}\Bigg)S_1+\frac{1}{(N-1)N^3(N+1)^3(N+2)^2}\frac{4P_{16}}{3}\zeta_3\Bigg]$$

$$+C_AC_FT_F\Bigg[-\frac{8P_{10}}{3(N-1)N^3(N+1)^3(N+2)^2}S_{2,1}+\frac{8P_{12}}{3(N-1)N^3(N+1)^3(N+2)^2}S_{-3}$$
$$+\frac{16P_{13}}{3(N-1)N^3(N+1)^3(N+2)^2}S_{-2,1}+\frac{8P_{22}}{27(N-1)^2N^3(N+1)^3(N+2)^2}S_3$$
$$-\frac{4P_{24}}{27(N-1)^2N^4(N+1)^4(N+2)^3}S_1^2-\frac{4P_{26}}{27(N-1)^2N^4(N+1)^4(N+2)^3}S_2$$
$$-\frac{8P_{33}}{243(N-1)^2N^6(N+1)^6(N+2)^5}+\Bigg(\frac{4P_4}{27(N-1)N(N+1)(N+2)}S_1^3$$
$$+\Bigg(\frac{8}{9}(137N^2+137N+334)S_3-\frac{16}{3}(35N^2+35N+18)S_{-2,1}\Bigg)S_1$$
$$+\frac{8}{3}(69N^2+69N+94)S_{-3}S_1+\frac{64}{3}(7N^2+7N+13)S_{-2}S_2+\frac{2}{3}(29N^2+29N+74)S_2^2$$
$$+\frac{4}{3}(143N^2+143N+310)S_4-\frac{16}{3}(3N^2+3N-2)S_{-2}^2+\frac{16}{3}(31N^2+31N+50)S_{-4}$$
$$-8(7N^2+7N+26)S_{3,1}-64(3N^2+3N+2)S_{-2,2}-\frac{32}{3}(23N^2+23N+22)S_{-3,1}$$
$$+\frac{64}{3}(13N^2+13N+2)S_{-2,1,1}+\frac{4P_4}{3(N-1)N(N+1)(N+2)}S_1\zeta_2$$
$$-\frac{8}{3}(11N^2+11N+10)S_1\zeta_3\Bigg)G+\Bigg(\frac{112}{3}S_{-2}S_1^2+\frac{2}{9}S_1^4+\frac{68}{3}S_1^2S_2-\frac{80}{3}S_{2,1,1}$$
$$+32\Bigg(\Bigg(-S_3(2)+S_{1,2}(2,1)-S_{2,1}(2,1)+S_{1,1,1}(2,1,1)\Bigg)S_1\left(\frac{1}{2}\right)+S_{1,3}\left(2,\frac{1}{2}\right)$$
$$-S_{2,2}\left(2,\frac{1}{2}\right)+S_{3,1}\left(2,\frac{1}{2}\right)-S_{1,1,2}\left(2,\frac{1}{2},1\right)+S_{1,1,2}\left(2,1,\frac{1}{2}\right)+S_{1,2,1}\left(2,\frac{1}{2},1\right)$$
$$-S_{1,2,1}\left(2,1,\frac{1}{2}\right)+S_{2,1,1}\left(2,\frac{1}{2},1\right)+S_{2,1,1}\left(2,1,\frac{1}{2}\right)-S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)$$
$$-S_{1,1,1,1}\left(2,1,\frac{1}{2},1\right)-S_{1,1,1,1}\left(2,1,1,\frac{1}{2}\right)\Bigg)\Bigg)+\Big(4S_1^2+12S_2+24S_{-2}\Big)\zeta_2$$
$$+224S_1\left(\frac{1}{2}\right)\zeta_3-144\zeta_4+16B_4\Bigg)F+\frac{16P_2 2^{-N}}{(N-1)N^3(N+1)^2}\Bigg(S_3(2)-S_{1,2}(2,1)$$
$$+S_{2,1}(2,1)-S_{1,1,1}(2,1,1)-7\zeta_3\Bigg)+\Bigg(\frac{4P_{11}}{9(N-1)^2N^3(N+1)^3(N+2)^2}S_2$$
$$+\frac{4P_{30}}{81(N-1)^2N^5(N+1)^5(N+2)^4}\Bigg)S_1+\Bigg(\frac{32P_6}{3(N-1)N^3(N+1)^3(N+2)^2}S_1$$
$$-\frac{8P_{18}}{3(N-1)N^4(N+1)^4(N+2)^3}\Bigg)S_{-2}-\frac{4P_{25}}{9(N-1)^2N^4(N+1)^4(N+2)^3}\zeta_2$$
$$-\frac{8P_{20}}{9(N-1)^2N^3(N+1)^3(N+2)^2}\zeta_3\Bigg].$$

[Ablinger et al. '14]

Example for a result: $a_{Qq}^{\mathrm{PS},(3)}$

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$$C_F T_F^2 \Bigg[\frac{32}{27(N-1)(N+3)(N+4)(N+5)} \Bigg(\frac{P_{15}}{N^3(N+1)^2(N+2)^2} S_2 - \frac{P_{19}}{N^3(N+1)^3(N+2)^2} S_1^2 + \frac{2P_{28}}{3N^4(N+1)^4(N+2)^3} S_1 - \frac{2P_{32}}{9N^5(N+1)^4(N+2)^4} \Bigg)$$
$$- \frac{32P_3}{9(N-1)N^3(N+1)^2(N+2)^2}\zeta_2 + \Bigg(\frac{32}{27}S_1^3 - \frac{160}{9}S_1S_2 - \frac{512}{27}S_3 + \frac{128}{3}S_{2,1} \ldots$$
$$+ \frac{32}{3}S_1\zeta_2 - \frac{1024}{9}\zeta_3 \Bigg) F \Bigg] + C_F N_F T_F^2 \Bigg[\frac{16P_7}{27(N-1)N^3(N+1)^3(N+2)^2} S_1^2$$
$$+ \frac{208P_7}{27(N-1)N^3(N+1)^3(N+2)^2} S_2 - \frac{32P_{21}}{81(N-1)N^4(N+1)^4(N+2)^3} S_1 + \ldots$$

$$+ C_A C_F T_F \Bigg[- \frac{8P_{10}}{3(N-1)N^3(N+1)^3(N+2)^2} S_{2,1} + \frac{8P_{12}}{3(N-1)N^3(N+1)^3(N+2)^2} S_{-3}$$
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$$- \frac{8P_{33}}{243(N-1)^2N^6(N+1)^6(N+2)^5} + \Bigg(\frac{4P_4}{27(N-1)N(N+1)(N+2)} S_1^3$$
$$+ \Bigg(\frac{8}{9}(137N^2+137N+334)S_3 - \frac{16}{3}(35N^2+35N+18) S_{-2,1} \Bigg) S_1$$
$$+ \frac{8}{3}(69N^2+69N+94)S_{-3}S_1 + \frac{64}{3}(7N^2+7N+13)S_{-2}S_2 + \frac{2}{3}(29N^2+29N+74)S_2^2 \ldots$$

$S_{-2,1}$

Harmonic Sums

e.g. $$S_{-2,1}(N) = \sum_{i=1}^{N} \frac{(-1)^i}{i^2} \sum_{j=1}^{i} \frac{1}{j}$$

- Typical example for a nested sum
- Sums like this appeared in perturbative QCD since its early days
- Until recently sufficient for completed OMEs

$$\ldots + \frac{8P_{27}}{3(N-1)N^5(N+1)^5(N+2)^4} \Bigg) S_1 + \frac{1}{(N-1)N^3(N+1)^3(N+2)^2} \ldots \frac{4P_{16}}{3}\zeta_3 \Bigg] \quad \ldots - \frac{\ldots P_{20}}{9(N-1)^2N^3(N+1)^3(N+2)^2}\zeta_3 \Bigg].$$

[Ablinger et al. '14]

Example for a result: $a_{Qq}^{\mathrm{PS},(3)}$

$a_{Qq}^{(3),\mathrm{PS}}(N) =$

Generalized Harmonic Sums

e.g. $$S_{1,3}\left(2,\frac{1}{2};N\right) = \sum_{i=1}^{N}\frac{2^i}{i}\sum_{j=1}^{i}\frac{(1/2)^j}{j^3}$$

- More complicated nested sum
- Additional parameter (e.g. 2^i) in the summand

$$+\frac{4P_{31}}{3(N-1)N^6(N+1)^6(N+2)^4} + \left(\left(\frac{2P_5}{N^2(N+1)^2} - \frac{4P_1}{N(N+1)}S_1\right)\zeta_2\right.$$
$$\left.-\frac{4P_1}{9N(N+1)}S_1^3\right)G + \left(\left(\frac{80}{9}S_3 - 64S_{2,1}\right)S_1 - \frac{2}{9}S_1^4 - \frac{20}{3}S_1^2S_2 + \frac{46}{3}S_2^2 + \frac{124}{3}S_4\right.$$
$$+\frac{416}{3}S_{2,1,1} + 64\left(\left(S_3(2) - S_{1,2}(2,1) + S_{2,1}(2,1) - S_{1,1,1}(2,1,1)\right)S_1\left(\frac{1}{2}\right)\right.$$
$$-S_{1,3}\left(2,\frac{1}{2}\right) + S_{2,2}\left(2,\frac{1}{2}\right) - S_{3,1}\left(2,\frac{1}{2}\right) + S_{1,1,2}\left(2,\frac{1}{2},1\right) - S_{1,1,2}\left(2,1,\frac{1}{2}\right)$$
$$-S_{1,2,1}\left(2,\frac{1}{2},1\right) + S_{1,2,1}\left(2,1,\frac{1}{2}\right) - S_{2,1,\ldots}$$
$$+S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right) + S_{1,1,1,1}\left(2,1,\frac{1}{2},1\right) \ldots - 4S_1^2\Big)\zeta_2$$
$$+\left(\frac{112}{3}S_1 - 448S_1\left(\frac{1}{2}\right)\right)\zeta_3 + 144\zeta_4 - 32B_4\Big)F + \frac{32P_2 2^{-N}}{(N-1)N^3(N+1)^2}\Big(-S_3(2)$$
$$+S_{1,2}(2,1) - S_{2,1}(2,1) + S_{1,1,1}(2,1,1) + 7\zeta_3\Big) + \left(-\frac{4P_8}{3(N-1)N^3(N+1)^3(N+2)^2}S_2\right.$$
$$\left.+\frac{8P_{27}}{3(N-1)N^5(N+1)^5(N+2)^4}\right)S_1 + \frac{1}{(N-1)N^3(N+1)^3(N+2)^2}\frac{4P_{16}}{3}\zeta_3\Bigg]$$

$$S_{1,3}\left(2,\frac{1}{2}\right)$$

$$+C_AC_FT_F\left[-\frac{8P_{10}}{3(N-1)N^3(N+1)^3(N+2)^2}S_{2,1} + \frac{8P_{12}}{3(N-1)N^3(N+1)^3(N+2)^2}S_{-3}\right.$$
$$\ldots S_3 \quad \ldots S_2 \quad \ldots + 29N + 74)S_2^2 \quad \ldots 31N + 50)S_{-4} \quad \ldots + 22)S_{-3,1} \quad \ldots S_{2,1,1}$$
$$+32\left(\left(-S_3(2) + S_{1,2}(2,1) - S_{2,1}(2,1) + S_{1,1,1}(2,1,1)\right)S_1\left(\frac{1}{2}\right) + S_{1,3}\left(2,\frac{1}{2}\right)\right.$$
$$-S_{2,2}\left(2,\frac{1}{2}\right) + S_{3,1}\left(2,\frac{1}{2}\right) - S_{1,1,2}\left(2,\frac{1}{2},1\right) + S_{1,1,2}\left(2,1,\frac{1}{2}\right) + S_{1,2,1}\left(2,\frac{1}{2},1\right)$$
$$-S_{1,2,1}\left(2,1,\frac{1}{2}\right) + S_{2,1,1}\left(2,\frac{1}{2},1\right) + S_{2,1,1}\left(2,1,\frac{1}{2}\right) - S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)$$
$$\left.-S_{1,1,1,1}\left(2,1,\frac{1}{2},1\right) - S_{1,1,1,1}\left(2,1,1,\frac{1}{2}\right)\right) + \left(4S_1^2 + 12S_2 + 24S_{-2}\right)\zeta_2$$
$$+224S_1\left(\frac{1}{2}\right)\zeta_3 - 144\zeta_4 + 16B_4\Big)F + \frac{16P_2 2^{-N}}{(N-1)N^3(N+1)^2}\Big(S_3(2) - S_{1,2}(2,1)$$
$$+S_{2,1}(2,1) - S_{1,1,1}(2,1,1) - 7\zeta_3\Big) + \left(\frac{4P_{11}}{9(N-1)^2N^3(N+1)^3(N+2)^2}S_2\right.$$
$$\left.+\frac{4P_{30}}{81(N-1)^2N^5(N+1)^5(N+2)^4}\right)S_1 + \left(\frac{32P_6}{3(N-1)N^3(N+1)^3(N+2)^2}S_1\right.$$
$$\left.-\frac{8P_{18}}{3(N-1)N^4(N+1)^4(N+2)^3}\right)S_{-2} - \frac{4P_{25}}{9(N-1)^2N^4(N+1)^4(N+2)^3}\zeta_2$$
$$\left.-\frac{8P_{20}}{9(N-1)^2N^3(N+1)^3(N+2)^2}\zeta_3\right].$$

[Ablinger et al. '14]

Example for a result: $a_{Qq}^{\mathrm{PS},(3)}$

$$a_{Qq}^{(3),\mathrm{PS}}(N) = C_F T_F^2 \Bigg[\frac{32}{27(N-1)(N+3)(N+4)(N+5)} \Bigg(\frac{P_{15}}{N^3(N+1)^2(N+2)^2} S_2 - \frac{P_{19}}{N^3(N+1)^3(N+2)^2} S_1^2 + \frac{2P_{28}}{3N^4(N+1)^4(N+2)^3} S_1 - \frac{2P_{32}}{9N^5(N+1)^4(N+2)^4} \Bigg) - \frac{32P_3}{9(N-1)N^3(N+1)^2(N+2)^2}\zeta_2 + \Bigg(\frac{32}{27}S_1^3 - \frac{160}{9}S_1S_2 - \frac{512}{27}S_3 + \frac{128}{3}S_{2,1} + \frac{32}{3}S_1\zeta_2 - \frac{1024}{9}\zeta_3\Bigg)F\Bigg] + C_F N_F T_F^2 \Bigg[\frac{16P_7}{27(N-1)N^3(N+1)^3(N+2)^2}S_1^2 + \frac{208P_7}{27(N-1)N^3(N+1)^3(N+2)^2}S_2 - \frac{32P_{21}}{81(N-1)N^4(N+1)^4(N+2)^3}S_1 + \ldots$$

$$+ C_A C_F T_F \Bigg[-\frac{8P_{10}}{3(N-1)N^3(N+1)^3(N+2)^2}S_{2,1} + \frac{8P_{12}}{3(N-1)N^3(N+1)^3(N+2)^2}S_{-3} + \frac{16P_{13}}{3(N-1)N^3(N+1)^3(N+2)^2}S_{-2,1} + \frac{8P_{22}}{27(N-1)^2N^3(N+1)^3(N+2)^2}S_3 - \frac{4P_{24}}{27(N-1)^2N^4(N+1)^4(N+2)^3}S_1^2 - \frac{4P_{26}}{27(N-1)^2N^4(N+1)^4(N+2)^3}S_2 - \frac{8P_{33}}{243(N-1)^2N^6(N+1)^6(N+2)^5} + \Bigg(\frac{4P_4}{27(N-1)N(N+1)(N+2)}S_1^3 + \Bigg(\frac{8}{9}(137N^2+137N+334)S_3 - \frac{16}{3}(35N^2+35N+18)S_{-2,1}\Bigg)S_1 + \frac{8}{3}(69N^2+69N+94)S_{-3}S_1 + \frac{64}{3}(7N^2+7N+13)S_{-2}S_2 + \frac{2}{3}(29N^2+29N+74)S_2^2 + \ldots$$

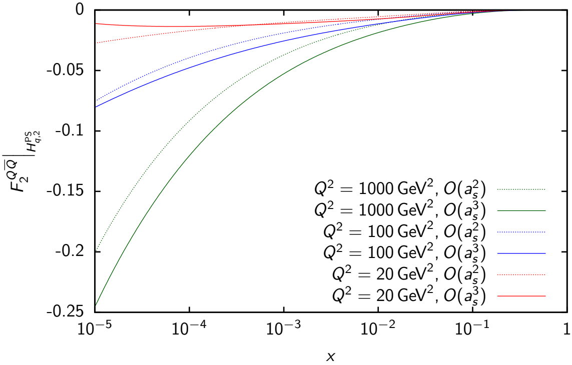
Binomially Weighted Sums

e.g. $$\sum_{i_1=1}^{N} \frac{\sum_{i_2=1}^{i_1} \frac{\binom{2i_2}{i_2} S_{1,1}(i_2)}{i_2}}{\binom{2i_1}{i_1}(1+2i_1)}$$

- Even more complicated nested sum
- Now also binomial coefficients $\binom{2i}{i}$ the summand
- Appear in other recent results ($A_{gg,Q}$ and A_{Qg})

[Ablinger et al. '14]

Unpolarized pure-singlet Wilson coefficient $H_{q,2}^{\mathrm{PS}}$

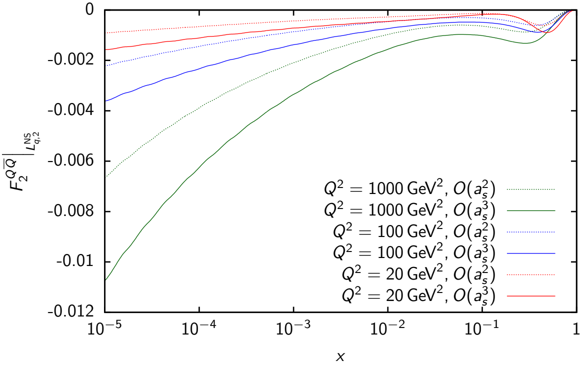


[Ablinger et al. '14]

For comparison (HERA kinematics): [Alekhin, Blümlein, Moch '13]

x	10^{-4}	10^{-3}	10^{-2}	10^{-1}
$F_2(x, Q^2 = 20\,\mathrm{GeV}^2)$	1.94	1.14	0.64	0.40
$F_2(x, Q^2 = 100\,\mathrm{GeV}^2)$		1.70	0.83	0.41
$F_2(x, Q^2 = 1000\,\mathrm{GeV}^2)$			1.04	0.41

Unpolarized non-singlet Wilson coefficient $L_{q,2}^{\mathrm{NS}}$



[Ablinger et al. '14]

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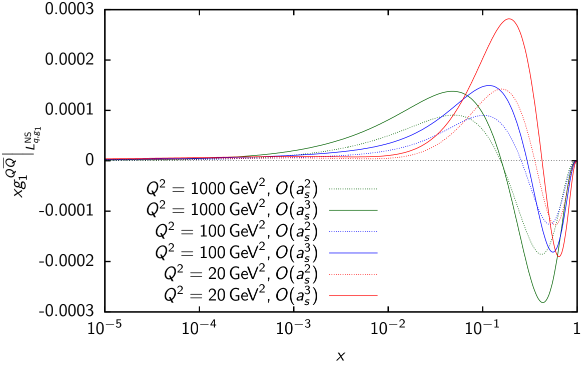
Polarized non-singlet Wilson coefficient L_{q,g_1}^{NS}

[Behring et al. '15]



Variable flavor number scheme (VFNS)

- Transition from scheme with n_f massless and 1 massive flavor to scheme with $n_f + 1$ effectively massless flavors
- Relevant for PDFs at the LHC
- Massive OMEs appear in the matching conditions of the PDFs

NNLO matching condition for non-singlet case:
[Buza et al. '96] [Bierenbaum, Blümlein, Klein, '09, '09]

$$\begin{aligned} & f_k(n_f+1,\mu^2) + \bar{f}_k(n_f+1,\mu^2) \\ & \quad = A_{qq,Q}^{\mathrm{NS}} \otimes [f_k(n_f,\mu^2) + \bar{f}_k(n_f,\mu^2)] \\ & \quad + \frac{1}{n_f}\left(A_{qq,Q}^{\mathrm{PS}} \otimes \Sigma(n_f,\mu^2) + A_{qg,Q} \otimes G(n_f,\mu^2)\right) \end{aligned}$$

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&+ \frac{1}{n_f}\left(A_{qq,Q}^{\mathrm{PS}} \otimes \Sigma(n_f,\mu^2) + A_{qg,Q} \otimes G(n_f,\mu^2)\right)
\end{aligned}
$$

- Ingredients at NNLO are now complete for the above relation
- $A_{qq,Q}^{\mathrm{PS}}$ and $A_{qg,Q}$ start at NNLO

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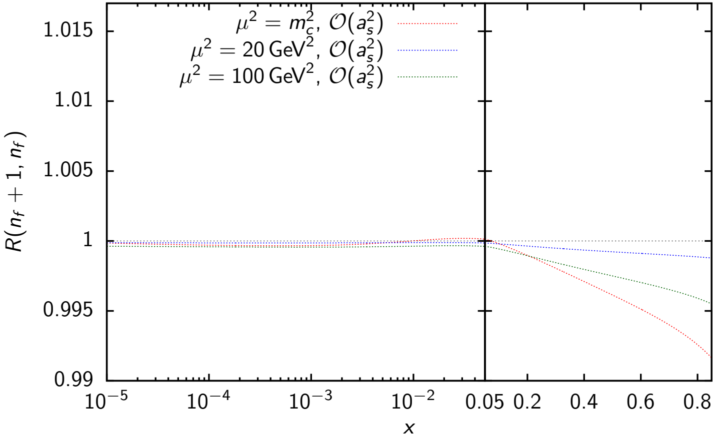

[Ablinger et al. '14]

$$R(n_f+1, n_f) = \frac{u(n_f+1, \mu^2) + \bar{u}(n_f+1, \mu^2)}{u(n_f, \mu^2) + \bar{u}(n_f, \mu^2)}, \quad \text{here } n_f = 3$$

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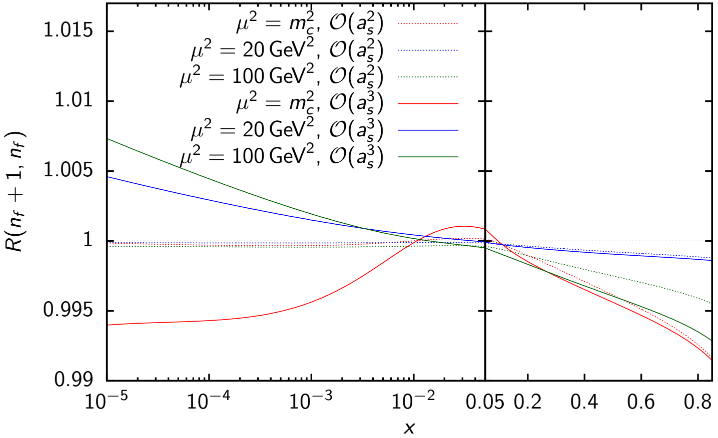

[Ablinger et al. '14]

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Conclusions

- Heavy quark corrections yield important contributions to DIS
  $\rightarrow$ essential for precision measurements
  of α_s (1%) and m_c (3%). [Alekhin et al. '12]
- New mathematical and computer-algebraic methods required for analytic calculation of the 3-loop corrections
  $\rightarrow$ includes new classes of higher transcendental functions and function spaces
- Completed Wilson coefficients and massive OMEs:
  - $L_{q,2}^{\mathrm{PS}}$, $L_{g,2}^{\mathrm{S}}$, $L_{q,2}^{\mathrm{NS}}$, $H_{q,2}^{\mathrm{PS}}$, L_{q,g_1}^{NS}
  - $A_{qq,Q}^{\mathrm{PS}}$, $A_{qg,Q}$, $A_{qq,Q}^{\mathrm{NS}}$, $A_{qq,Q}^{\mathrm{TR}}$, A_{Qq}^{PS}, $A_{gq,Q}$, $A_{gg,Q}$
- Ingredients for first matching relation of the VFNS are complete.
- Calculation of the remaining massive OME A_{Qg}
  and Wilson coefficient $H_{g,2}^{\mathrm{S}}$ is in progress.