Diagrammatic representation of the coproduct of one-loop Feynman diagrams

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Diagrammatic interpretation of the coproduct

The coproduct of one-loop (scalar) Feynman diagrams has a completely diagrammatic representation. Schematically,

$$\Delta(F) = \sum_i L_i \otimes R_i$$

- F is a Feynman diagram with n propagators ;
- L_i are Feynman diagrams with $m \leq n$ propagators ;
- R_i are cuts of the diagram F.

Valid in dimensional regularisation to all orders in ϵ.

Allows to bypass the need of integration by parts (IBP) relations to get differential equations and show that coefficients of differential equations are derivatives of cuts.

$$F = \frac{e^{\gamma_E \epsilon}}{\pi^{\frac{D}{2}}} \int d^D k \prod_{j=1}^{n} \frac{1}{q_j^2 - m_j^2 + i0}$$

$$q_j = \alpha_j k + \sum_{l=1}^{n} \beta_{jk} q_l, \quad \alpha_j, \beta_{jk} \in \{-1, 0, 1\}$$

We choose $D = d - 2\epsilon$ with $d \in \mathbb{N}$, even, such that $d - 2 < n \leq d$. E.g.:

- tadpoles and bubbles: $D = 2 - 2\epsilon$;
- triangles and boxes: $D = 4 - 2\epsilon$;
- pentagons and hexagons: $D = 6 - 2\epsilon$;
- $\ldots$;

F is a function of weight $d/2$

Cuts of Feynman diagrams

Cuts solve Landau equations[1], and capture **discontinuities** across (physical) branch cuts:

- of amplitudes ;

  'The Analytic S-Matrix', optical theorem, dispersion relations[2], modern unitarity methods
- of individual Feynman diagrams.

  Largest Time Equation, dispersion relations for Feynman diagrams [3]

In practice, computed using

$$\frac{1}{k^2 - m^2 \pm i0} \rightarrow \theta\left(k_0\right)\delta\left(k^2 - m^2\right)$$

[1] Landau (1959), Cutkosky (1960)

[2] R. J. Eden, P. V. Landshoff, D. I. Olive, J. C. Polkinghorne (1966)

[3] *Diagrammar*, G. 't Hooft and M. Veltman (1973); E. Remiddi (1982)

Multiple Polylogarithms:

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t) \qquad a_i, z \in \mathbb{C}$$

ex: $G\left(\vec{0}_n; z\right) = \frac{1}{n!}\log^n z\,;$ $\qquad G\left(\vec{0}_{n-1}, a, ; z\right) = -\text{Li}_n\left(\frac{z}{a}\right)$

A large class of Feynman diagrams can be written in terms of MPL.

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Transcendental weight:

$$w\left(G(a_1,\ldots,a_n;z)\right) = n \qquad w\left(\zeta_n\right) = n \qquad w\left(\pi^n\right) = n$$

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$\mathbb{Q}$-vector space of MPL forms Hopf algebra (graded by weight) — $\mathcal{H}$

Equipped with a **coproduct** $\Delta:\mathcal{H}\to\mathcal{H}\otimes\mathcal{H}$

ex: $\Delta_{1,1}\left(\log^2(z)\right)=2\log(z)\otimes\log(z)\,;$ $\qquad \Delta_{1,2}\left(\mathrm{Li}_3(z)\right)=-\frac{1}{2}\log(1-z)\otimes\log^2(z)$

Coproduct and discontinuities

$$\Delta \mathrm{Disc} = (\mathrm{Disc} \otimes \mathrm{id})\, \Delta$$

Discontinuities act on the **first entry** of the coproduct

Coproduct and differential operators

$$\Delta \frac{\partial}{\partial z} = \left(\mathrm{id} \otimes \frac{\partial}{\partial z}\right) \Delta$$

Differential operators act on the **last entry** of the coproduct

$$\Delta(F) = \sum_i L_i \otimes R_i$$

Case 1: R_i is a cut of m propagators with m **odd**.

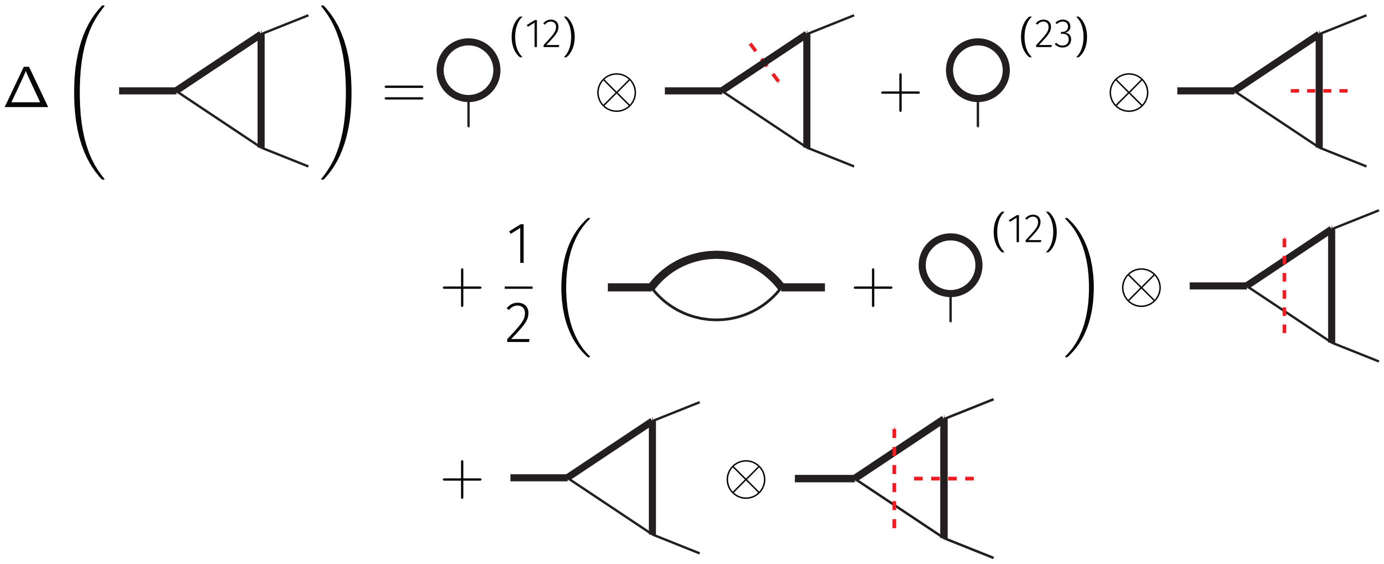
L_i is a diagram with m propagators obtained by deleting the uncut propagators.

Case 2: R_i is a cut of m propagators with m **even**.

L_i is a sum of diagrams (times 1/2):

- the diagram with m propagators obtained by deleting the uncut propagators ;
- all diagrams with $m-1$ propagators obtained by deleting one more propagator.

EXAMPLE: $T(p_1^2; m_{12}^2, m_{23}^2)$



- Valid to all orders in ϵ.
- Non-trivial cancelation of bubble and tadpole divergences.
- Correctly reproduces all components of coproduct $\Delta_{n,m}$.

Differential equations to compute Feynman integrals

E.Remiddi, *Il Nuovo Cimento A series 11,* 1997; T.Gehrmann and E.Remiddi, *Nucl.Phys.B.* 580, 2000

- Take derivative w.r.t. scale (internal mass or external channel),

$$\text{e.g.}\quad \frac{\partial}{\partial m^2}\left(\frac{1}{q^2-m^2}\right)=\frac{1}{\left(q^2-m^2\right)^2}$$

$\Rightarrow$ Get linear combination of diagrams with propagators raised to different powers.

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- Use IBP relations to reduce all diagrams that were generated by taking derivatives to a set of 'master integrals' f_i.

  ⇒ Solve large system of equations (FIRE, REDUZE, ...).

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$$\frac{\partial}{\partial m^2}F=\sum c\left(\{s_i\},\{m_i\},\epsilon\right)f_i$$

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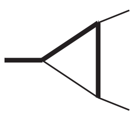
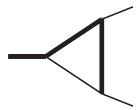
$$\frac{\partial}{\partial m^2} F = \sum c\left(\{s_i\}, \{m_i\}, \epsilon\right) f_i$$

- Example:

$\frac{\partial}{\partial \mu_{12}} \Big($  $\Big) = -\frac{1}{2(1+\mu_{23}-\mu_{12})}$  $-\left(\frac{1}{2(1+\mu_{23}-\mu_{12})} + \frac{1}{\mu_{12}-\mu_{23}}\right)$  $^{(12)}$

$+\left(\frac{1}{1+\mu_{23}-\mu_{12}} + \frac{1}{\mu_{12}-\mu_{23}}\right)$  $^{(23)}$ $+ \frac{\epsilon}{1+\mu_{23}-\mu_{12}}$ 

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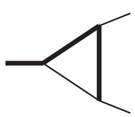
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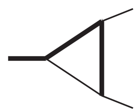
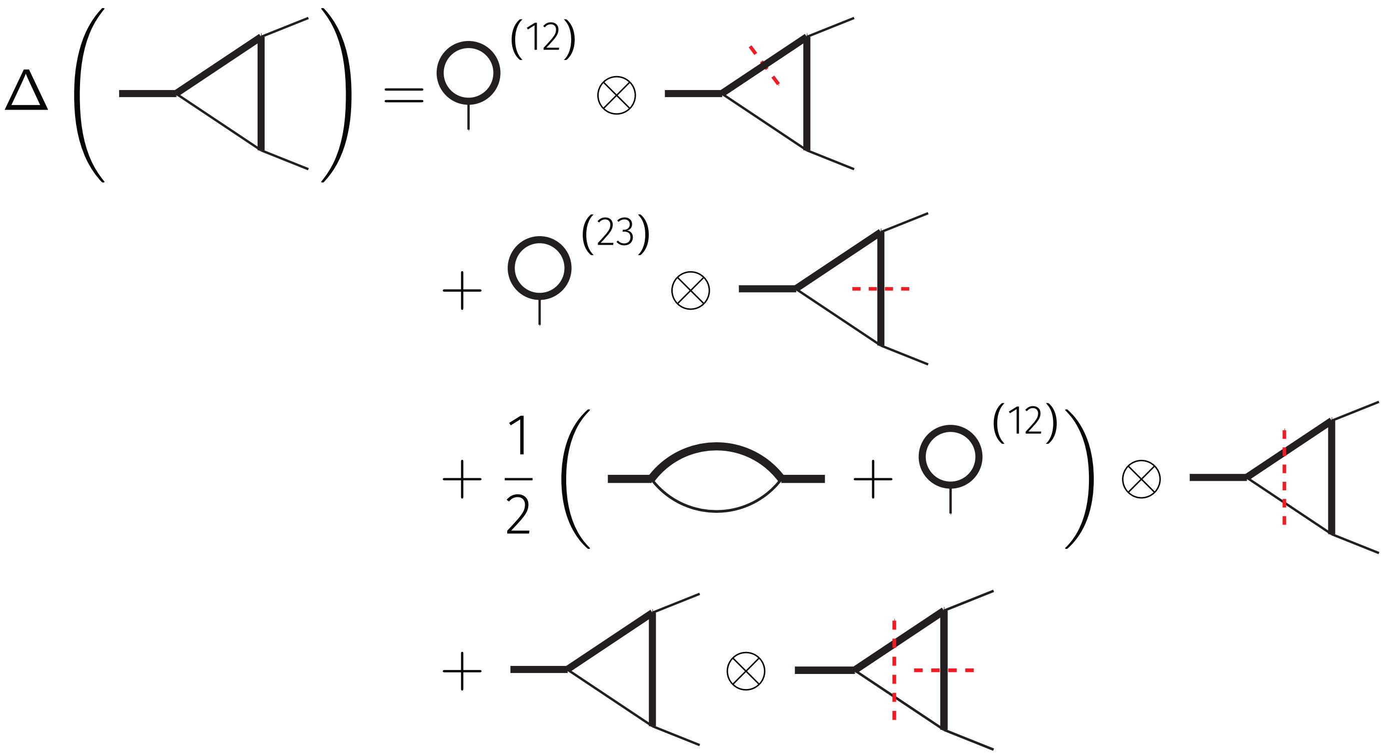


$c\left(\{s_i\},\{m_i\},\epsilon\right)$ are derivatives of cuts!

$$\Delta \frac{\partial}{\partial z} = \left(\mathrm{id} \otimes \frac{\partial}{\partial z}\right) \Delta$$

Example: Differential equation of $T(p_1^2; m_{12}^2, m_{23}^2)$ with $\mu_{12} = m_{12}^2/p_1^2$



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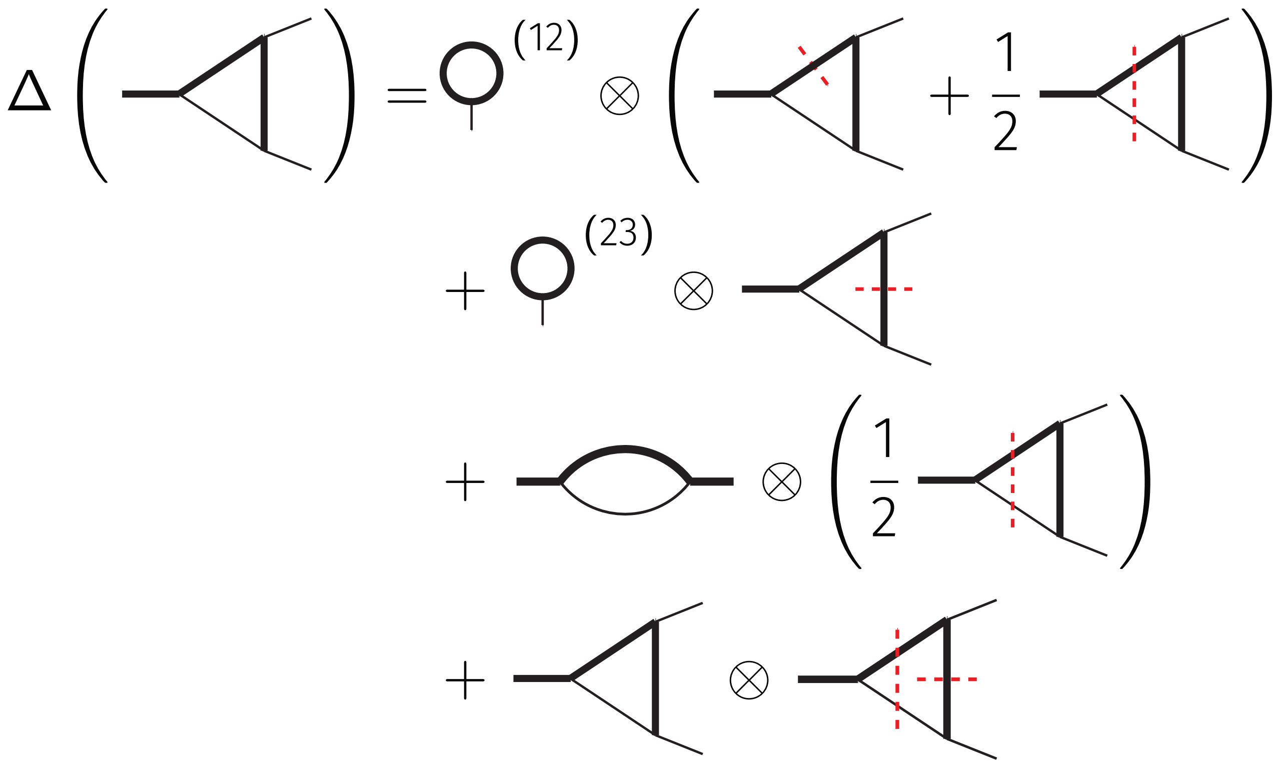

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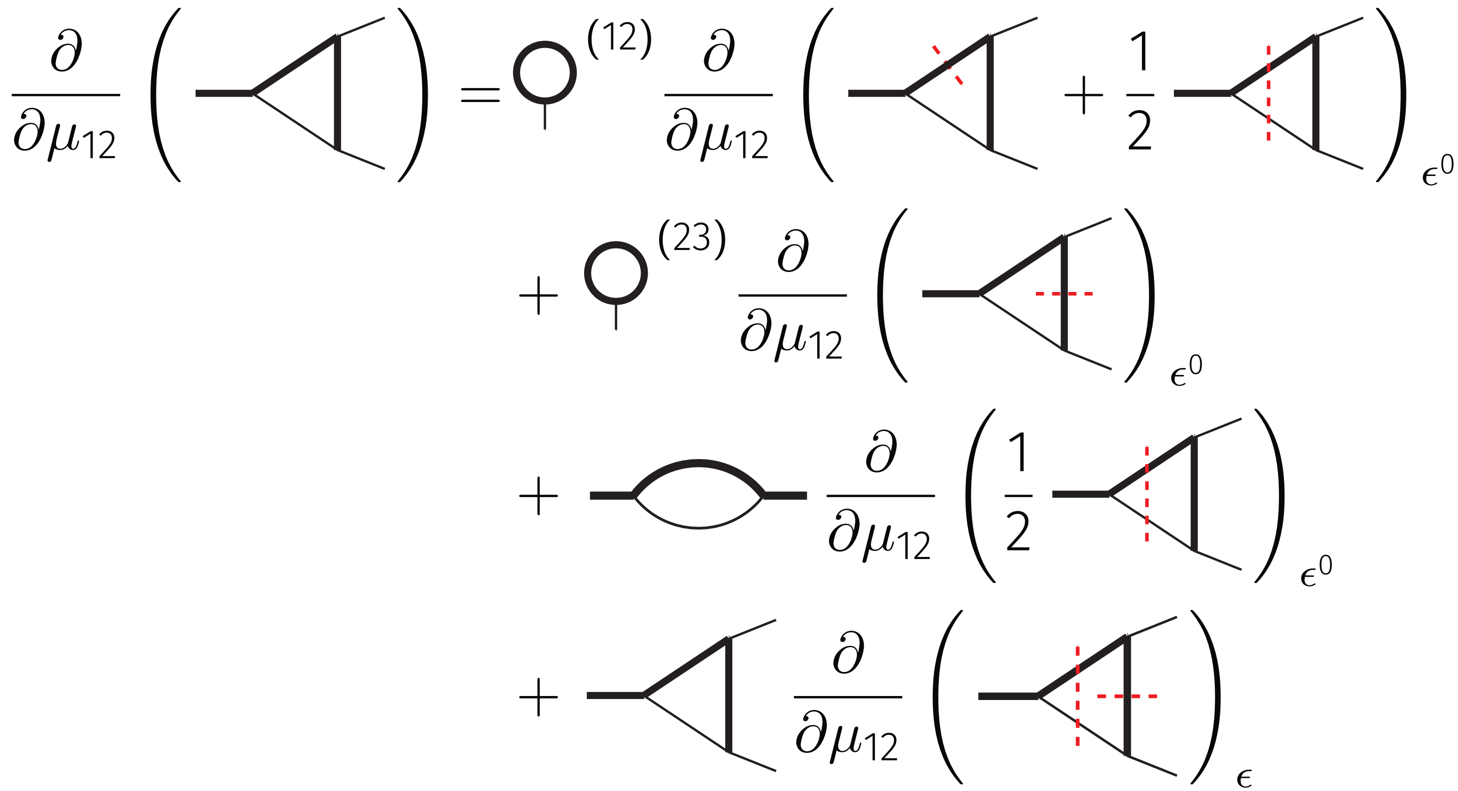

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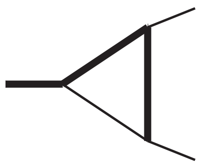
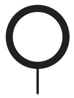
$\frac{\partial}{\partial \mu_{12}}$ () =  $^{(12)} \left(-\frac{1}{2(1+\mu_{23}-\mu_{12})} - \frac{1}{\mu_{12}-\mu_{23}} \right)$

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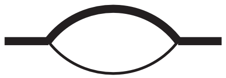
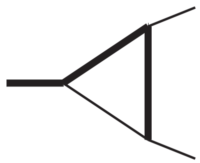
$+$  $\left(-\frac{1}{2(1+\mu_{23}-\mu_{12})} \right)$

$+$  $\left(\frac{\epsilon}{1+\mu_{23}-\mu_{12}} \right)$

Outlook

Can our construction be generalised to two and more loops?

- In which dimensions should diagrams be evaluated?
- Which combinations of diagrams appear in the first entry?

Can our construction be generalised to diagrams not expressible in terms of MPLs?

- We only use the fact that diagrams are expressible as MPLs in the check of the conjecture, our diagrammatic rules could be more general.

Thank you!