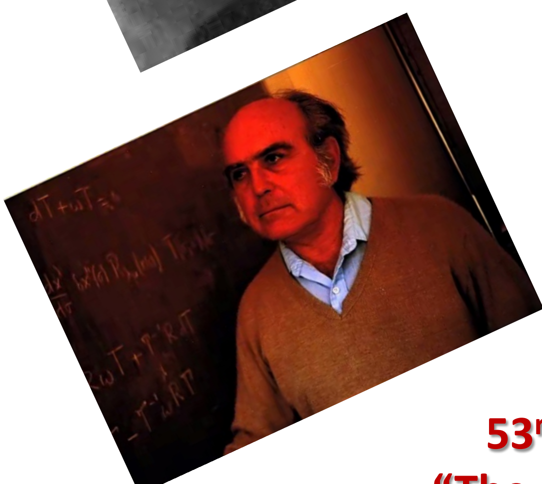
Majorana fermions
Supersymmetry Breaking
Born-Infeld Theory

Sergio FERRARA

(CERN)



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Plan of the Lectures

a. **Majorana fermions**

b. **Spontaneous Supersymmetry Breaking**

c. **Born-Infeld Theory and Duality**

Part a

Majorana fermions

Supersymmetry deals with basic spinorial entities called **“Majorana spinors”**. This notion originates from a fundamental observation of the Italian (Sicilian) physicist Ettore Majorana, who noticed that if the Dirac γ-matrices are suitably chosen, the Dirac equation admits ”real” solutions

*“Teoria symmetrica dell’elettrone e del positrone”, Nuovo Cimento **14** (1937) 171*

This is the so-called **Majorana representation**, where the γ-matrices are all real. Therefore the Dirac equation

$$(\gamma^\mu \partial_\mu + m) \psi = 0$$

admits manifestly real solutions.

In the Majorana representation the γ_i are symmetric, while γ_0 and $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$ are antisymmetric, and satisfy the Clifford algebra relations

$$\{\gamma_\mu, \gamma_\nu\} = 2\,\eta_{\mu\nu} \qquad \eta_{\mu\nu} = (-,+,+,+)$$

How about chiral spinors? They cannot be real, since they are eigenstates of γ_5, corresponding to eigenvalues $\pm\, i$ (γ_5^2 = -1) :

$$\gamma_5\,\psi_\pm \;=\; \pm i\,\psi_\pm$$

So $$\psi_\pm \;=\; \frac{1}{2}(1 \mp i\,\gamma_5)\,\psi \qquad \psi_+ + \psi_- = \psi \qquad \psi_+^\star = \psi_-$$

Spinors which are eigenstates of γ_5 are called Weyl spinors.

From the preceding properties it follows that Weyl spinors have two components instead of four, and are complex, in D=4.

Symmetries of γ's depend on the dimension D=s+t of space time, and reality properties on its signature ρ=s-t (modulo 8: Bott periodicity). For ordinary Minkowski space ρ=D-2. For example, in D=10 the Majorana representation is valid for both ψ_+ and ψ_- , since γ_{11}^2 = 1. For a comprehensive treatment of spinors and Clifford algebra for arbitrary (s,t) see, for instance:

R. D'Auria, SF, M.A. Lledo and V.S. Varadarajan, J. Geom. Phys. ***40*** *(2001) 101*

The massless Dirac equation has an extra symmetry, which in the Majorana representation takes the form

$$\psi' = e^{\alpha\,\gamma_5}\,\psi$$

This is indeed a U(1) rotation, since $\gamma_5{}^2 = -1$.

However, this symmetry is broken by the mass term for a Majorana fermion (called **Majorana mass**). Defining a complex Dirac spinor

$$\psi_D = \psi_1 + i\,\psi_2\,, \quad \psi_D^\star = \psi_1 - i\,\psi_2\,, \quad (\psi_1, \psi_2 \text{ Majorana})$$

one can now have a U(1) symmetry which rotates ψ_1 into ψ_2 .

We can now define:

$$\psi_L = \frac{1}{2}\,(1 - i\,\gamma_5)\,\psi_D\,, \quad \psi_R = \psi_L^\star = \frac{1}{2}\,(1 + i\,\gamma_5)\,\psi_D^\star$$

$$\chi_L = \frac{1}{2}\,(1 - i\,\gamma_5)\,\psi_D^\star\,, \quad \chi_R = \chi_L^\star = \frac{1}{2}\,(1 + i\,\gamma_5)\,\psi_D$$

Therefore ψ_L,χ_L (ψ_R ,χ_R) have opposite U(1) phases, while ψ_L,χ_R (ψ_R,χ_L) have identical U(1) phases.

The U(1) invariant Dirac equation in the chiral (Weyl) notation becomes

$$\gamma^\mu \partial_\mu \psi_L + m \chi_R = 0 \,, \quad \gamma^\mu \partial_\mu \chi_L + m \psi_R = 0 \quad (+\, h.c.)$$

Therefore the Dirac mass term, in this notation, is $m \psi_L \chi_L + h.c.$, which is of course U(1) invariant.

In principle, if we give up the U(1) symmetry, **a Dirac fermion can have three types of mass terms**,

$$m \psi_L \chi_L \,, \quad M \psi_L \psi_L \,, \quad N \chi_L \chi_L$$

m: Dirac mass ; (M,N): Majorana masses

As a result, the matrix

$$\mathcal{M} = \begin{pmatrix} M & m \\ m & N \end{pmatrix}$$

has the two eigenvalues,

$$m_{1,2} = \frac{1}{2}\left[M + N \pm \sqrt{(M-N)^2 + 4\,m^2}\right]$$

In particular, for $N = 0$, $M >> m$ the eigenvalues become

$$m_{1,2} = \left(M\,,\, -\frac{m^2}{M}\right)$$

This is at the basis of the **see-saw mechanism.**

Minkowski; Gell-Mann, Ramond, Slansky

REALITY PROPERTIES OF SPINORS

IN A SPACE-TIME WITH (s,t) SIGNATURE, $D=s+t$, $\rho=s-t$

EIGHT CASES (BOTT PERIODICITY)

COMPLEX SPINORS $\rho=2,6$

REAL SPINORS $\rho=0,1,7$

PSEUDOREAL SPINORS $\rho=3,4,5$

IN MINKOWSKI SPACE-TIME $(D-1,1)$ $\rho=D-2$

D even

COMPLEX SPINORS $D=4,8$

REAL SPINORS $D=2,\underline{10}$ IT APPLIES TO $S_{\pm}$

PSEUDOREAL SPINORS $D=6$

D odd

REAL SPINORS $D=3,9,\underline{11}$

PSEUDO REAL SPINORS $D=5,7$

A SPINOR IS REAL IF $\psi^* = \psi$

A SPINOR IS PSEUDOREAL IF $\psi^{*I}_\alpha = \Omega^{IJ} C_{\alpha\beta} \psi^J_\beta$

$(\Omega^2 = -1,\ C^2 = -1)$

Weyl spinors are real for $D = 2 \bmod 8$

Weyl spinors are pseudoreal for $D = 6 \bmod 8$

Weyl spinors are complex for $D = 4, 8 \bmod 8$

The automorphism group of the SUPERSYMMETRY ALGEBRA is in these cases

$SO(N_+) \times SO(N_-)$ $\quad D = 2 \bmod 8$

$Sp(2N_+) \times Sp(2N_-)$ $\quad D = 6 \bmod 8$

$U(N)$ $\quad D = 4, 8 \bmod 8$

SUPERSYMMETRY ALGEBRAS (WITHOUT CENTRAL EXTENSION)

$$\{ Q_\alpha^A , Q_\beta^B \} = \Omega^{AB} (\gamma^\mu)_{\alpha\beta} P_\mu + \ldots$$

$D = 5, 6, 7$

$$\{ Q_\alpha^A , Q_{\dot\alpha B} \} = \delta^A_B (\gamma^\mu)_{\alpha\dot\alpha} P_\mu + \ldots$$

$D = 4, 8$

$$\{ Q_\alpha^A , Q_\beta^B \} = \delta^{AB} (\gamma^\mu)_{\alpha\beta} P_\mu + \ldots$$

$D = 3, 9, 10, 11$

11 D (M THEORY) ALGEBRA (WITH CENTRAL EXTENSION)

$$\{ Q_\alpha , Q_\beta \} = (\gamma^\mu)_{\alpha\beta} P_\mu + (\gamma^{\mu\nu})_{\alpha\beta} Z_{\mu\nu} + (\gamma^{\mu\nu\rho\sigma\delta})_{\alpha\beta} Z_{\mu\nu\rho\sigma\delta}$$

points two brane five brane

COUNTING COMPONENTS :

D even $\quad S_\pm$ spinors : $2^{\frac{D-2}{2}}$ $\quad$ D even (twice for complex or pseudoreal)

D odd $\quad S$ spinors : $2^{\frac{D-1}{2}}$ $\quad$ D odd (twice for pseudoreal)

For example : D even $\quad D=4 \quad S_+ \to 2_C \;,\; D=6 \quad S_+ = 4_C \;,\; D=10 \quad S_+ = 16_R$

$D=8 \quad S_+ \to 8_C \quad (D=2 \quad S_+ \to 1_R)$

D odd : $D=3 \quad S \to 2_R \;,\; D=5 \quad S \to 4_C \;,\; D=7 \quad S \to 8_C \;,\; D=9 \quad S \to 16_R$

$D=11 \quad S \to 32_R$

8 REAL SPINOR COMPONENTS ARE POSSIBLE IF $D \leq 6$

16 REAL SPINOR COMPONENTS ARE POSSIBLE IF $D \leq 10$

32 REAL SPINOR COMPONENTS ARE POSSIBLE IF $D \leq 11$

GRAVITINO : Spin 3/2

$\psi_{\mu\alpha}$ (vector-spinor)

MASSLESS (IR of SO(D-2)) SO ITS COMPONENTS ARE $(D-2)\times 2^{\frac{D-3}{2}} - 2^{\frac{D-3}{2}} = (D-3)\, 2^{\frac{D-3}{2}}$ D odd, S real

$D=11 \rightarrow 8\times 16 = 128$

$D=10 \rightarrow (D-3)\, 2^{\frac{D-4}{2}} = 7\times 8 = 56_{\pm}$

$$\mathcal{L}_{SG} = R - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho$$

F

W. NAHM

POINCARÉ SUPERSYMMETRY + SUPERGRAVITY $\rightarrow$ $D \leq 11$

CONFORMAL SUPERSYMMETRY $\rightarrow$ $D \leq 6$

BASED ON CLASSIFICATION OF SUPERALGEBRAS

(KATZ)

SYMMETRY PROPERTIES OF SPINORS

(EIGHT CASES, BOTT PERIODICITY)

	K even (symmetry)	K odd
D even		
D = 8	$S^{\pm} \otimes S^{\pm} \to \Lambda^K \quad (-1)^{K(K-1)/2}$	$S^{\pm} \otimes S^{\mp} \to \Lambda^K$
D = 2	$S^{\pm} \otimes S^{\mp} \to \Lambda^K$	$S^{\pm} \otimes S^{\pm} \to \Lambda^K \quad (-1)^{K(K-1)/2}$
D = 4	$S^{\pm} \otimes S^{\pm} \to \Lambda^K \quad -(-)^{K(K-1)/2}$	$S^{\pm} \otimes S^{\mp} \to \Lambda^K$
D = 6	$S^{\pm} \otimes S^{\mp} \to \Lambda^K$	$S^{\pm} \otimes S^{\pm} \to \Lambda^K \quad -(-)^{K(K-1)/2}$

EXAMPLE $K=1$ $\quad \gamma^\mu P_\mu$ (symmetric $D=2,10\ldots$ antisymmetric $D=6$)

D odd	K even (symmetry)		K odd	
D=1	$S\otimes S\to\Lambda^K$	$(-)^{K(K-1)/2}$	$S\otimes S\to\Lambda^K$	$(-)^{K(K-1)/2}$
D=3	$S\otimes S\to\Lambda^K$	$-(-)^{K(K-1)/2}$	$S\otimes S\to\Lambda^K$	$(-)^{K(K-1)/2}$
D=5	$S\otimes S\to\Lambda^K$	$-(-)^{K(K-1)/2}$	$S\otimes S\to\Lambda^K$	$-(-)^{K(K-1)/2}$
D=7	$S\otimes S\to\Lambda^K$	$(-)^{K(K-1)/2}$	$S\otimes S\to\Lambda^K$	$-(-)^{K(K-1)/2}$

EXAMPLE $K=1$ $\gamma^\mu P_\mu$ (symmetric $D=1,3$ antisymmetric $D=5,7$)

Part b

Spontaneous Supersymmetry Breaking

In supersymmetric theories there are conserved **Majorana vector-spinor** currents, the Noether currents of Supersymmetry (in van der Waerden notation):

$$\partial^\mu J^A_{\mu\,\alpha}(x) \;=\; 0 \quad (A = 1, .., N)$$

The corresponding charges

$$Q^A_\alpha \;=\; \int d^3\mathbf{x}\, J^A_{0\,\alpha}(x)$$

generate the Supersymmetry algebra

$$\{Q^A_\alpha\,,\,\overline{Q}_{\dot\alpha\;B}\} \;=\; (\sigma_\mu)_{\alpha\dot\alpha}\,P^\mu\;\delta^A_B\;, \quad \{Q^A_\alpha\,,\,Q^B_\beta\} \;=\; 0$$

p-branes can modify this algebra by adding "central extension" terms:

(Townsend; SF, Porrati)

$$\{Q^A_\alpha\,,\,\overline{Q}_{\dot\alpha\;B}\} \;=\; (\sigma_\mu)_{\alpha\dot\alpha}\,P^\mu\;\delta^A_B \;+\; Z^{A\,\mu}_{B}\;(\sigma_\mu)_{\alpha\dot\alpha} \quad \text{(strings)}$$

$$\{Q^A_\alpha\,,\,Q^B_\beta\} \;=\; Z^{[AB]}\,\epsilon_{\alpha\beta} \;+\; Z^{(AB)}_{(\alpha\beta)} \quad \text{(points, membranes)}$$

(Dvali, Shifman; Polchinski; Bagger, Galperin; Rocek, Tseytlin)

In the N=1 spontaneous breaking, the order parameter f enters a term linear in the supercurrents (dim(f)=2 in nat. units):

$$J_{\mu\,\dot\alpha} \;=\; f\;(\gamma_\mu G)_{\dot\alpha} + \;\ldots$$

G_α is a Majorana fermion, the **goldstino**.

The SUSY current algebra implies

$$\int d^3\mathbf{y}\,\{J_{0\,\dot\alpha}(y), J_{\mu\,\alpha}(x)\} \;=\; (\sigma^\nu)_{\alpha\,\dot\alpha}\; T_{\nu\mu}(x) \;+\; \text{derivatives}$$

so that

$$\langle 0|\,\{\overline{Q}_{\dot\alpha}, J_{\mu\alpha}(x)\}\,|\rangle \;=\; (\sigma^\nu)_{\alpha\,\dot\alpha}\;\langle T_{\nu\mu}\rangle \;=\; (\sigma_\mu)_{\alpha\,\dot\alpha}\;\; f^2$$

and for several charges

$$\langle 0|\,\{\overline{Q}_{\dot\alpha\,A}, J^B_{\mu\alpha}(x)\}\,|\rangle \;=\; (\sigma^\nu)_{\alpha\,\dot\alpha}\;\langle 0|T_{\nu\mu}|0\rangle\,\delta^B_A \;=\; (\sigma_\mu)_{\alpha\,\dot\alpha}\;\; f^2\,\delta^B_A$$

So without central extension and in the rigid case (not Supergravity), either all Supersymmetries are unbroken or they are all broken at the same scale f.

$$J_{\mu\,\dot\alpha\,A} \;=\; f\;(\gamma_\mu G_A)_{\dot\alpha} + \;...$$ *(Witten)*

One can invalidate this result modifying the current algebra via terms not proportional to δ_A^B , adding a contribution proportional to C_A^B , a constant matrix in the adjoint of SU(N), *i.e.* traceless and Hermitian.

$$\langle 0|\left\{\overline{Q}_{\dot\alpha\,A}, J^B_{\mu\alpha}(x)\right\}|0\rangle \;=\; (\sigma^\nu)_{\alpha\,\dot\alpha}\;\langle 0|T_{\nu\mu}|0\rangle\,\delta_A^B \;+\; (\sigma_\mu)_{\alpha\,\dot\alpha}\;C_A^B$$

Effective goldstino actions depend on both N and C_A^B , since it can be shown that

$$\delta_A\chi^i\,\delta^B\overline{\chi}_i \;=\; V\,\delta_A^B \;+\; C_A^B \qquad \left(\delta\chi^i \;=\; \delta_A\chi^i\,\epsilon^A\right)$$

If k supersymmetries are unbroken, the (Hermitian) matrix $V\mathbf{1} + C$ will have rank N-k in the vacuum. The N-1 possible scales of partial supersymmetry breaking will be classified by the N-1 Casimir operators of SU(N). Amusingly, the characteristic equation can be solved algebraic form up to quartic order, which corresponds to the N=4 case.

Field theoretical examples are known for N=2

(Antoniadis, Partouche, Taylor; SF, Porrati, Sagnotti)

and the previous analysis in terms of SU(2) invariants was performed in *(Andrianopoli, D'Auria, SF, Trigiante).* In this case the SU(2) quadratic invariant is the squared norm of a 3-vector ξ^x constructed via electric and magnetic Fayet-Iliopoulos terms

$$\xi^x = (\mathcal{Q}_y \wedge \mathcal{Q}_z)\,\epsilon^{xyz} \ , \quad \mathcal{Q}_y \wedge \mathcal{Q}_z = m_y^\Lambda\, e_{z\Lambda} \ - \ m_z^\Lambda\, e_{y\Lambda}$$

(Λ=1,…,n), with n the number of vector multiplets.

The Hermitian matrix

$$\delta_A \chi^i \, \delta^B \overline{\chi}_i \;=\; \begin{pmatrix} V - \xi^3 & \xi^1 + i\,\xi^2 \\ \xi^1 - i\,\xi^2 & V + \xi^3 \end{pmatrix}$$

has eigenvalues

$$\lambda_{1,2} = V \;\mp\; \sqrt{\xi^x\,\xi^x}$$

When $V \;=\sqrt{\xi^x\,\xi^x}\,,\;\lambda_1$ vanishes and N=2 is broken to N=1 with $f_2 \;=\; \left(\xi^x\,\xi^x\right)^{\frac{1}{4}}\,.$

The goldstino action for N=1 and for general N spontaneously broken to N=0 ($C_A^B = 0$) was derived by Volkov and Akulov in the N=1 case, and in general reads

$$\mathcal{L}_{VA}\left(f, G^A_\alpha\right) \;=\; f^2\left[1 \;-\; \sqrt{-\det\left(\eta_{\mu\nu} \;+\; \frac{i}{f^2}\;\left(\overline{G}_A\gamma_\mu\partial_\nu\, G^A - h.c.\right)\right)}\right]$$

The Goldstino Lagrangian contains a finite number of terms because of the nilpotency of the G^A . This action has U(N) symmetry (R-symmetry) and it is invariant under the following transformation:

$$\delta G_\alpha^A(x) = f\,\epsilon_\alpha^A + \frac{i}{f}\left(\overline{G}_B\,\gamma^\mu\epsilon^B - \bar{\epsilon}_B\,\gamma^\mu G^B\right)\,\partial_\mu G_\alpha^A$$

The Goldstino action for partially broken N=2 supersymmetry is instead described by the supersymmetric Born-Infeld action

(Deser, Puzalowski; Cecotti, SF; Bagger, Galperin; Rocek, Tseytlin; Kuzenko, Theisen)

$\mathcal{L}\left(G_\alpha, F_{\mu\nu}\right)$, with the properties

$$\mathcal{L}\left(G_\alpha, F_{\mu\nu} = 0\right) \longrightarrow \mathcal{L}_{VA}$$

$$\mathcal{L}\left(G_\alpha = 0, F_{\mu\nu}\right) = f^2\left[1 - \sqrt{-\,det\left(\eta_{\mu\nu} + \frac{1}{f}\,F_{\mu\nu}\right)}\right]$$

The gaugino of N=1 is the goldstino of the second broken Supersymmetry.

In terms of microscopic parameters, the scale of the broken supersymmetry is an SU(2) and symplectic invariant

$$f = (\xi^x\, \xi^x)^{\frac{1}{4}} \ \sim \ (e_2\, m_1)^{\frac{1}{2}} \qquad (\text{for n} = 1, \text{m}_2 = 0, \mathcal{Q}_3 = 0)$$

The Goldstino action for a partial breaking N->N-k is only known for N=2, which corresponds to the non-linear limit of the quadratic action described before. In the simplest case of an N=2 vector multiplet, composed of an N=1 vector multiplet and an N=1 chiral multiplet

$$\left(1, 2\left(\frac{1}{2}\right), 0, 0\right) \longrightarrow \left(1, \frac{1}{2}\right) + \left(\frac{1}{2}, 0, 0\right)$$

$m_V = 0 \qquad m_x \neq 0$

The C_A^B allow to give a mass to the chiral multiplet. In the large mass limit the chiral multiplet can be "integrated out", giving rise to a non-linear theory for the fields $(G_\alpha, F_{\mu\nu})$.

$$\text{Chiral Multiplet} \quad X \;:\; (\overline{D}_{\dot\alpha} X = 0) \;\longrightarrow\; \left(\frac{1}{2}, 0, 0\right)$$

$$\text{Vector Multiplet} \quad W_\alpha \;:\; (\overline{D}_{\dot\alpha} W_\alpha = 0) \;\longrightarrow\; \left(1, \frac{1}{2}\right)$$

Superfield constraint : (Bagger,Galperin)

$$X \;=\; -\,\frac{W^2}{m_1 \,-\, \overline{D}^2\,\overline{X}} \;\longrightarrow\; X \text{ non} - \text{linear function of } W_\alpha, \overline{W}_{\dot\alpha}$$

$$\mathcal{L}_{SBI} \;=\; Im \int d^2\theta (e_1 + i e_2)\, X \left(m_1, D^2 W^2, \overline{D}^2 \overline{W}^2\right)$$

$$f = \sqrt{e_2\, m_1}\;, \vartheta = \frac{e_1}{e_2} \quad \text{(for canonically normalized vectors)}$$

$$\mathcal{L}_{SBI} \;=\; \vartheta\, Im \int d^2\theta\, W^2 \;+\; Re \int d^2\theta\, W^2 \;-\; \frac{1}{f^4} \int d^4\theta\, W^2\, \overline{W}^2 \left[\left(X^2 - Y\right)^{\frac{1}{2}} \;+\; X\right]^{-1}$$

$$X = 1 \,-\, \frac{1}{2}\left(T + \overline{T}\right)\;, \quad Y = T\,\overline{T}\,, \quad T = \frac{1}{f^2}\, \overline{D}^2 \overline{W}^2\,, \quad T|_{\theta=0} \;=\; -\,\frac{1}{2\,f^2}\, F_+^2$$

$$F_{\pm,\mu\nu} \;=\; \frac{1}{2}\left(F_{\mu\nu} \;\pm\; i\,\widetilde{F}_{\mu\nu}\right)\,, \quad F_+^\star \;=\; F_-$$

(Cecotti, SF)

Part c

Born-Infeld Theory and Duality

Electric-magnetic duality is one of the most fascinating symmetries of non-linear theories.

Free Maxwell theory is the prototype of a duality invariant theory. The relativistic form of its equations in vacuum is

$$\partial_\mu F^{\mu\nu} = 0 \qquad \Box A_\mu - \partial_\mu \partial \cdot A = 0$$
$$\partial_\mu \widetilde{F}^{\mu\nu} = 0 \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Letting $F_{\mu\nu} = \left(\vec{E}, \vec{B}\right)$ one recovers the conventional form

$$\partial_t \vec{E} = \nabla \times \vec{B} \qquad \nabla \cdot \vec{B} = 0$$
$$\partial_t \vec{B} = -\nabla \times \vec{E} \qquad \nabla \cdot \vec{E} = 0$$
$$(F_{0i} = E_i \,, \quad F_{ij} = \epsilon_{ijk} B_k)$$

The **energy-momentum tensor** is

$$T_{\mu\nu}: \quad T_{ij} = E_i\, E_j \;+\; B_i\, B_j \;, \quad T_{0i} \;=\; \epsilon_{ijk} E_j B_k \;, \quad T_{00} \;= \vec{E}^2 \;+\; \vec{B}^2$$

and **the equations of motion**, together with $T_{\mu\nu}$ (and in particular the Hamiltonian) **are invariant under U(1) rotations**

$$\begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix}' = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix}$$

$$\delta_\alpha\, T_{\mu\nu} \;=\; 0$$

Note that the Lagrangian $\mathcal{L} \;=\; \frac{1}{2}\left(\vec{E}^2 \;-\; \vec{B}^2\right)$ is not invariant.

Moreover ***e.m.* duality is not an internal symmetry, since it rotates a tensor into a pseudo-tensor** (a sort of bosonic chiral transformation)

Non-linear theories of electromagnetism (with possible addition of other matter fields and electric and/or magnetic sources) are obtained introducing an **"electric displacement"** $\vec{D}$ and a **"magnetic field"** $\vec{H}$ that only for the free Maxwell theory coincide with the **electric field** $\vec{E}$ and the **magnetic induction** $\vec{B}$

$$\vec{E} = \vec{D}\,, \qquad \vec{B} = \vec{H}$$

In a medium (with possible sources) the non—linear *e.m.* equations become

$$\partial_t\vec{B} + \nabla\times\vec{E} = 0\left(4\pi\vec{J}_m\right) \qquad \nabla\cdot\vec{B} = 0\,(4\pi\rho_m)$$

$$-\,\partial_t\vec{D} + \nabla\times\vec{H} = 4\pi\vec{J}_e \qquad \nabla\cdot\vec{D} = 4\pi\rho_e$$

where $\left(\vec{E},\vec{B},\vec{D},\vec{H}\right)$ are linked by the **constitutive relations**

$$\vec{D} = \vec{D}\left(\vec{E}\,,\,\vec{B}\right) \qquad \vec{H} = \vec{H}\left(\vec{E}\,,\,\vec{B}\right)$$

with $\vec{D} = \vec{E} + \ldots, \quad \vec{H} = \vec{B} + \ldots$

we observe that **the non-linear equations are invariant under rotations**

$$\left(\vec{B}, -\vec{D}\right) \to \left(\vec{B}, -\vec{D}\right)_\alpha \qquad \left(\vec{E}, \vec{H}\right) \to \left(\vec{E}, \vec{H}\right)_\alpha$$

In a relativistic notation one can write

$$F_{\mu\nu} = \left(\vec{E}, \vec{B}\right) \qquad G_{\mu\nu} = \left(\vec{H}, -\vec{D}\right)$$

and the non-linear equations with sources become

$$\partial_\mu \widetilde{G}^{\mu\nu} = J_e^\nu , \qquad \partial_\mu \widetilde{F}^{\mu\nu} = 0\,(J_m^\nu)$$

and are invariant under rotations $(F_{\mu\nu}\ G_{\mu\nu}) \to (F_{\mu\nu}\,,\ G_{\mu\nu})_\alpha$ provided sources rotate accordingly.

Here $G_{\mu\nu} = G_{\mu\nu}(F_{\mu\nu})$, and in the vacuum $F_{\mu\nu} = -\widetilde{G}_{\mu\nu}$

The **source-free stress tensor** now becomes *(Gaillard and Zumino, 1981)*

$$T^{\mu}{}_{\lambda} \;=\; -\,\partial_\lambda\,\varphi^i\;\frac{\partial\mathcal{L}}{\partial\partial_\mu\,\varphi^i}\;+\;\delta^{\mu}_{\lambda}\;\mathcal{L}\;+\;\widetilde{G}^{\mu\nu}\;F_{\nu\lambda}$$

(when matter fields φ^i are included)

$T^{\mu}{}_{\lambda}$ **is invariant under duality rotations also acting on matter fields, and the first term is absent in a pure non-linear theory for** $F_{\mu\nu}$

Generalizing to **n Maxwell fields** is straightforward, and the original U(1) rotations become at most **U(n) rotations** :

$$\delta F \;=\; a\cdot F \;+\; b\cdot G$$
$$\delta G \;=\; c\cdot F \;+\; d\cdot G$$

with matrices a, b, c, d such that

$$d \;=\; a \;=\; -\,a^T\;,\quad b\;=\;b^T\;,\quad c\;=\;-\,b$$

The above invariance under electric-magnetic duality is far from trivial, since **the (F,G) rotation must be consistent with the constitutive relation** $G_{\mu\nu} = G_{\mu\nu}(F_{\mu\nu})$, and it is highly non trivial to find functionals that satisfy such compatibility constraints.

In addition, the request of having a Lagrangian formulation for such theories demands that

$$\widetilde{G}_{\mu\nu} = 2\,\frac{\partial \mathcal{L}}{\partial F^{\mu\nu}}$$

or, in non-covariant form

$$\vec{D} = 2\,\frac{\partial \mathcal{L}\left(\vec{E},\vec{B}\right)}{\partial \vec{E}} \qquad \vec{H} = -\,2\,\frac{\partial \mathcal{L}\left(\vec{E},\vec{B}\right)}{\partial \vec{E}}$$

with the integrability condition

$$\frac{\partial \widetilde{G}^{\mu\nu}}{\partial F^{\rho\sigma}} = \frac{\partial \widetilde{G}_{\rho\sigma}}{\partial F_{\mu\nu}}$$

It can be shown that **invariance under U(n) rotations requires that the constitutive relations satisfy the conditions**

(Gaillard and Zumino)

$$G^{\Lambda}\,\widetilde{G}^{\Sigma} \;+\; F^{\Lambda}\,\widetilde{F}^{\Sigma} \;=\; 0$$
$$G^{\Lambda}\,\widetilde{F}^{\Sigma} \;-\; G^{\Sigma}\,\widetilde{F}^{\Lambda} \;=\; 0$$

In particular, **for n=1 there is just one condition**

$$G\,\widetilde{G} \;+\; F\,\widetilde{F} \;=\; 0$$

which is trivially satisfied in Maxwell theory, where $F \;=\; -\,\widetilde{G}$

A non-trivial solution of the constraints is provided by the **Born-Infeld Theory** ***("Foundations of the new field theory", Proc. R. Soc. London A144 (1934) 425)*** (later reconsidered by Schrödinger, Dirac and many others), which takes the form

$$\mathcal{L}_{BI} \;=\; \mu^2 \left[\, 1 \;-\; \sqrt{-det\left(\eta_{\mu\nu} \;+\; \frac{1}{\mu}\,F_{\mu\nu}\right)}\,\right]$$
$$=\; \mu^2 \left[\, 1 \;-\; \sqrt{1 \;+\; \frac{1}{2\,\mu^2}\,F^2 \;-\; \frac{1}{16\,\mu^4}\,\left(F\cdot\widetilde{F}\right)^2}\,\right] \;=\; \mathcal{L}_{BI}\left(\vec{B}^2 \;-\; \vec{E}^2, \vec{E}\cdot\vec{B}\right)$$

Note the analogy with the relativistic γ factor for the velocity:

$$\mathcal{L}_{BI}\left(\vec{B}=0\right) \;=\; \mu^2\left[\,1\;-\;\sqrt{1\;-\;\frac{1}{\mu^2}\;\vec{E}^2}\,\right]$$

which sets an upper bound for $\vec{E}: \quad \left|\vec{E}\right| \;<\; \mu$

This Lagrangian has in fact unique properties against instabilities created by the medium. Not many generalizations are known in the multi-field case.

The link to modern theories comes from **higher-order curvature terms** produced by loop corrections or by α' corrections in **Superstring Theory**.

In addition, **in Supergravity electric-magnetic duality is a common feature in the Einstein approximation**, and it is natural to enforce it when the theory is deformed by higher (Einstein or Maxwell) curvature terms *(SF, Scherk, Zumino, Cremmer, Julia, Kallosh, Stelle, Bossard, Howe, Nicolai, …)*

The Born-Infeld action emerges in brane dynamics as the bosonic sector of the Goldstino action for N=2 supersymmetry spontaneously broken to N=1.

In this setting the **Maxwell field is the partner of the spin – ½ goldstino** in an N=1 vector multiplet.

(Cecotti, SF, Deser, Puzalowski, Hughes, Polchinski, Liu, Bagger, Galperin, Rocek, Tseytlin, Kuzenko, Theisen, …)

A multi-field extension is possible, in the N=2 setting, starting from an N=2 vector multiplet where suitable Fayet-Iliopoulos terms are introduced to integrate out N=1 chiral multiplets

(Antoniadis, Partouche, Taylor)

For a single N=2 vector multiplet the Born-Infeld action emerges as a solution of the **quadratic constraint**

$$F_+^2 + \mathcal{F}\left(m - \overline{\mathcal{F}}\right) = 0$$

where F_+ is the self-dual curvature of the Maxwell field

$$F_{+,\,\mu\nu} = \frac{1}{2}\left(F_{\mu\nu} + i\,\widetilde{F}_{\mu\nu}\right)$$

and $\mathcal{F}$ is an auxiliary field such that

$$\mathcal{L}_{BI} = m\, Re\, \mathcal{F}$$

In the multi-field case **the generalization rests on the constraints**

(SF, Porrati, Sagnotti)

$$d_{ABC}\left[F_+^B\,F_+^C\;+\;\mathcal{F}^B\left(m^C\;-\;\overline{\mathcal{F}}^C\right)\right]\;=\;0$$

where the d_{ABC} are the coefficients of the **cubic term of the holomorphic prepotential of rigid special geometry**, whose classification rests on the singularity structure of cubic varieties.

(FPS + Stora, Yeranyan)

Further generalizations of electric-magnetic dualities can be obtained coupling gauge vectors to scalar fields, as it occurs in Supergravity, where they can be partners of the graviton (in $N \geq 4$ extended Supergravity) or additional matter (for $N \leq 4$)

The general situation was again studied by Gaillard and Zumino, who showed that, in the presence of scalars, the most general electric-magnetic duality rotation for n vector fields belongs to a subgroup of Sp(2n,R).

In the infinitesimal these transformations act as

$$\begin{aligned} \delta F &= a \cdot F + b \cdot G \\ \delta G &= c \cdot F - a^T \cdot G \\ b &= b^T , \quad c = c^T \end{aligned}$$

Under such transformations (which contain U(n) in the particular case $a = -a^T \ , \quad b = -c$), the Lagrangian transforms as

$$\delta \mathcal{L} = \frac{1}{4} \left(F\, c\, \widetilde{F} + G\, b\, \widetilde{G} \right)$$

Therefore, **when b≠0 (*i.e.* when the electric field transforms into the magnetic one) the duality *is not* an invariance of the Lagrangian, although *it is* an invariance of the stress tensor**. The invariance can only hold for b=0 (c $\neq$ 0 implies a total derivative term), so for matrices of the form

$$\begin{pmatrix} a & 0 \\ c & -a^T \end{pmatrix}$$

For n=1, one recovers a notorious example, the SL(2,R) duality of the dilaton-axion system coupled to a Maxwell field.

Notice that the duality symmetry must not necessarily be the full Sp(2n, R), but at least a subgroup possessing a 2n-dimensional symplectic representation R, *i.e.* such that $R \times R \supset 1_a$ (as is the case, for instance, for the **56** of $E_{7,7}$ in N=8 Supergravity).

I would like to conclude this talk discussing **other examples of non-linear (square-root) Born-Infeld type Lagrangians in D dimensions** that were suggested from alternative 4D goldstino actions studied by *Bagger, Galperin, Rocek, Tseytlin, Kuzenko, Theisen*

The first class of examples are **D-dimensional generalizations containing pairs of form field strengths** of degrees (p+1,D-p-1), gauge fields that couple to (p-1,D-p-3) branes). These Lagrangians generalize the D=4, p=2 case corresponding to a non-linear Lagrangian for a tensor multiplet regarded as an N=2 → N=1 goldstino multiplet in rigid Supersymmetry.

(Kuzenko, Theisen; SF, Sagnotti, Yeranyan)

These action read

$$\mathcal{L} = \mu^2 \left[1 - \sqrt{1 + \frac{1}{\mu^2} X - \frac{1}{\mu^4} Y^2} \right]$$

$$X = \star \left[H_{p+1} \wedge \star H_{p+1} + V_{D-p-1} \wedge \star V_{D-p-1} \right]$$

$$Y = \star \left[H_{p+1} \wedge V_{D-p-1} \right]$$

and have the property of being **doubly self-dual** under

$$V'_{D-p-1} = \widetilde{H}_{p+1} \,, \quad H'_{p+1} = \widetilde{V}_{D-p-1}$$

Moreover, after a single duality the action ends up with two forms of the same degree, in a manifestly U(1) invariant combination

$$\mathcal{L} = \mu^2 \left[1 - \sqrt{1 + \frac{W_{D-p-1} \cdot \bar{W}_{D-p-1}}{\mu^2} + \frac{\left(W_{D-p-1} \cdot \bar{W}_{D-p-1}\right)^2 - W^2_{D-p-1} \bar{W}^2_{D-p-1}}{4\,\mu^4}} \right]$$

which has only a $U(1)_{em}$ duality.

A complexification of the n=1 Born-Infeld action with an SU(2) duality is

$$\mathcal{L} = \mu^2 \left[1 - \sqrt{1 + \frac{F_{p+1} \cdot \overline{F}_{p+1}}{\mu^2} - \frac{(\star\left[F_{p+1} \wedge F_{p+1}\right])\ \left(\star\left[\overline{F}_{p+1} \wedge \overline{F}_{p+1}\right]\right)}{\mu^4}} \right]$$

Actions with the full U(n) duality group were proposed by *Aschieri, Brace, Morariu, Zumino* but they are not available in closed form even for n=2.

These non-linear actions can be made massive introducing Green-Schwarz terms *(SF, Sagnotti)* , *i.e.* couplings to another gauge field of the form

$$m H_{p+1} \wedge A_{D-p-1}$$

A Stueckelberg mechanism is then generated, and **the non-linear mass terms take the same form as the original non-linear curvature action**, whose (D-p-1)-form gauge field has been eaten to give mass to the original p-form.

The simplest example is the four-dimensional Born-Infeld action used to make an antisymmetric field $b_{\mu\nu}$ massive. The mass term comes exactly from the Born-Infeld term with $F_{\mu\nu}$ replaced by $m\, b_{\mu\nu}$

This type of mechanism could be relevant when the system is coupled to N=2 Supergravity, in which case one of the two gravitini would belong to a massive spin-3/2 multiplet.