$(spin\ 2) = (spin\ 1)^2$

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1.0 Basic idea

- Strong nuclear, Weak nuclear and Electromagnetic forces described by Yang-Mills gauge theory (non-abelian generalisation of Maxwell).

  Gluons, W, Z and photons have spin 1.

- Gravitational force described by Einstein's general relativity.

  Gravitons have spin 2.

- But maybe $GRAVITY = (YANG - MILLS)^2$

- If so, gravitational symmetries should follow from those of Yang-Mills

1.1 Gravity as square of Yang-Mills

- A recurring theme in attempts to understand the quantum theory of gravity and appears in several different forms:
- Closed states from products of open states and KLT relations in string theory [Kawai:1985, Siegel:1988],
- On-shell $D = 10$ Type IIA and IIB supergravity representations from on-shell $D = 10$ super Yang-Mills representations [Green:1987],
- Supergravity scattering amplitudes from those of super Yang-Mills in various dimensions, [Bern:2008, Bern: 2010, 2012,Bianchi:2008,Huang:2012,Cachazo:2013,Dolan:2013]
- Ambitwistor strings [Hodges:2011, Mason:2013, Geyer:2014]
- Vector theory of gravity [arXiv:0904.3155 [gr-qc] Anatoly A. Svidzinsky (Texas A-M)]

1.2 Local and global symmetries from Yang-Mills

- LOCAL SYMMETRIES: general covariance, local lorentz invariance, local supersymmtry, local p-form gauge invariance

  [arXiv:1408.4434
  A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy]

- GLOBAL SYMMETRIES eg $G = E_7$ in $D = 4, \mathcal{N} = 8$ supergravity

  [arXiv:1301.4176 arXiv:1312.6523 arXiv:1402.4649
  A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy]

- YANG-MILLS SPIN-OFF (interesting in its own right): Unified description of $(D = 3; \mathcal{N} = 1, 2, 4, 8)$, $(D = 4; \mathcal{N} = 1, 2, 4)$, $(D = 6; \mathcal{N} = 1, 2)$, $(D = 10; \mathcal{N} = 1)$ Yang-Mills in terms of a pair of division algebras $(\mathbb{A}_n, \mathbb{A}_{n\mathcal{N}})$, $n = D - 2$ [arXiv:1309.0546

  A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy]

- GENERALIZED SELF-MIRROR CONDITION AND VANISHING TRACE ANOMALIES

1.3 Product?

- Although much of the squaring literature invokes taking a product of left and right Yang-Mills fields

$$A_\mu(x)(L) \otimes A_\nu(x)(R)$$

it is hard to find a conventional field theory definition of the product. Where do the gauge indices go? Does it obey the Leibnitz rule

$$\partial_\mu(f \otimes g) = (\partial_\mu f) \otimes g + f \otimes (\partial_\mu g)$$

If not, why not?

1.4 Convolution

- Here we present a $G_L \times G_R$ product rule :

$$[A_\mu{}^i(L) \star \Phi_{ii'} \star A_\nu{}^{i'}(R)](x)$$

where $\Phi_{ii'}$ is the "spectator" bi-adjoint scalar field introduced by Hodges [Hodges:2011] and Cachazo *et al* [Cachazo:2013] and where $\star$ denotes a convolution

$$[f \star g](x) = \int d^4y f(y) g(x-y).$$

Note $f \star g = g \star f, (f \star g) \star h = f \star (g \star h)$, and, importantly obeys

$$\partial_\mu (f \star g) = (\partial_\mu f) \star g = f \star (\partial_\mu g)$$

and not Leibnitz

$$\partial_\mu (f \otimes g) = (\partial_\mu f) \otimes g + f \otimes (\partial_\mu g)$$

1.5 Gravity/Yang-Mills dictionary

For concreteness we focus on

- $\mathcal{N} = 1$ supergravity in $D = 4$, obtained by tensoring the $(4 + 4)$ off-shell $\mathcal{N}_L = 1$ Yang-Mills multiplet $(A_\mu(L), \chi(L), D(L))$ with the $(3 + 0)$ off-shell $\mathcal{N}_R = 0$ multiplet $A_\mu(R)$.
- Interestingly enough, this yields the new-minimal formulation of $\mathcal{N} = 1$ supergravity [Sohnius:1981] with its 12+12 multiplet $(h_{\mu\nu}, \psi_\mu, V_\mu, B_{\mu\nu})$
- The dictionary is,

$$\begin{aligned}
Z_{\mu\nu} \equiv h_{\mu\nu} + B_{\mu\nu} &= {A_\mu}^i(L) \quad \star \quad \Phi_{ii'} \quad \star \quad {A_\nu}^{i'}(R) \\
\psi_\nu &= \chi^i(L) \quad \star \quad \Phi_{ii'} \quad \star \quad {A_\nu}^{i'}(R) \\
V_\nu &= D^i(L) \quad \star \quad \Phi_{ii'} \quad \star \quad {A_\nu}^{i'}(R),
\end{aligned}$$

1.6 Yang-Mills symmetries

- The left supermultiplet is described by a vector superfield $V^i(L)$ transforming as

$$\delta V^i(L) = \Lambda^i(L) + \bar{\Lambda}^i(L) + f^i{}_{jk} V^j(L)\theta^k(L) + \delta_{(a,\lambda,\epsilon)} V^i(L).$$

Similarly the right Yang-Mills field ${A_\nu}^{i'}(R)$ transforms as

$$\delta {A_\nu}^{i'}(R) = \partial_\nu \sigma^{i'}(R) + f^{i'}{}_{j'k'} {A_\nu}^{j'}(R)\theta^{k'}(R) + \delta_{(a,\lambda)} {A_\nu}^{i'}(R).$$

and the spectator as

$$\delta \Phi_{ii'} = -f^j{}_{ik}\Phi_{ji'}\theta^k(L) - f^{j'}{}_{i'k'}\Phi_{ij'}\theta^{k'}(R) + \delta_a \Phi_{ii'}.$$

Plugging these into the dictionary gives the gravity transformation rules.

1.7 Gravitational symmetries

$$
\begin{aligned}
\delta Z_{\mu\nu} &= \partial_\nu \alpha_\mu(L) + \partial_\mu \alpha_\nu(R), \\
\delta \psi_\mu &= \partial_\mu \eta, \\
\delta V_\mu &= \partial_\mu \Lambda,
\end{aligned}
$$

where

$$
\begin{array}{ccccccc}
\alpha_\mu(L) & = & A_\mu{}^i(L) & \star & \Phi_{ii'} & \star & \sigma^{i'}(R), \\
\alpha_\nu(R) & = & \sigma^i(L) & \star & \Phi_{ii'} & \star & A_\nu{}^{i'}(R), \\
\eta & = & \chi^i(L) & \star & \Phi_{ii'} & \star & \sigma^{i'}(R), \\
\Lambda & = & D^i(L) & \star & \Phi_{ii'} & \star & \sigma^{i'}(R),
\end{array}
$$

illustrating how the local gravitational symmetries of general covariance, 2-form gauge invariance, local supersymmetry and local chiral symmetry follow from those of Yang-Mills.

We also recover from the dictionary the component supersymmetry variation of [Sohnius:1981],

$$\begin{aligned}\delta_\epsilon Z_{\mu\nu} &= -4i\bar{\epsilon}\gamma_\nu\psi_\mu,\\ \delta_\epsilon \psi_\mu &= -\tfrac{i}{4}\sigma^{k\lambda}\epsilon\partial_k g_{\lambda\mu} + \gamma_5\epsilon V_\mu\\ &\quad - \gamma_5\epsilon H_\mu - \tfrac{i}{2}\sigma_{\mu\nu}\gamma_5\epsilon H^\nu,\\ \delta_\epsilon V_\mu &= -\bar{\epsilon}\gamma_\mu\sigma^{\kappa\lambda}\gamma_5\partial_\kappa\psi_\lambda.\end{aligned}$$

1.9 Lorentz multiplet

New minimal supergravity also admits an off-shell Lorentz multiplet $(\Omega_{\mu ab}{}^{-}, \psi_{ab}, -2V_{ab}{}^{+})$ transforming as

$$\delta\mathcal{V}^{ab} = \Lambda^{ab} + \bar{\Lambda}^{ab} + \delta_{(a,\lambda,\epsilon)}\mathcal{V}^{ab}. \tag{1}$$

This may also be derived by tensoring the left Yang-Mills superfield $V^{i}(L)$ with the right Yang-Mills field strength $F^{abi'}(R)$ using the dictionary

$$\mathcal{V}^{ab} = V^{i}(L) \star \Phi_{ii'} \star F^{abi'}(R),$$

$$\Lambda^{ab} = \Lambda^{i}(L) \star \Phi_{ii'} \star F^{abi'}(R).$$

1.10 Bianchi identities

- The corresponding Riemann and Torsion tensors are given by

$$R^{+}_{\mu\nu\rho\sigma} = -F_{\mu\nu}{}^{i}(L) \star \Phi_{ii'} \star F_{\rho\sigma}{}^{i'}(R) = R^{-}_{\rho\sigma\mu\nu}.$$

$$T^{+}_{\mu\nu\rho} = -F_{[\mu\nu}{}^{i}(L) \star \Phi_{ii'} \star A_{\rho]}{}^{i'}(R) = -A_{[\rho}{}^{i}(L) \star \Phi_{ii'} \star F_{\mu\nu]}{}^{i'}(R) = -T^{-}_{\mu\nu\rho}$$

- One can show that (to linearised order) the gravitational Bianchi identities follow from those of Yang-Mills

$$D_{[\mu}(L)F_{\nu\rho]}{}^{I}(L) = 0 = D_{[\mu}(R)F_{\nu\rho]}{}^{I'}(R)$$

1.11 To do

- Convoluting the off-shell Yang-Mills multiplets $(4+4, \mathcal{N}_L = 1)$ and $(3+0, \mathcal{N}_R = 0)$ yields the $12+12$ new-minimal off-shell $\mathcal{N} = 1$ supergravity.
- We expect that convoluting the off-shell general multiplet $(8+8, \mathcal{N}_L = 1)$ and $(3+0, \mathcal{N}_R = 0)$ yields the $24+24$ non-minimal off-shell $\mathcal{N} = 1$ supergravity [Breitenlohner:1977].
- We expect convoluting $(4+4, \mathcal{N}_L = 1)$ and $(4+4, \mathcal{N}_R = 1)$ yields the $32+32$ minimal off-shell $\mathcal{N} = 2$ supergravity [Fradkin:1979, deWit:1979, Breitenlohner:1979, Breitenlohner:1980]. The latter would involve bosons from the product of left and right fermions.
- Clearly two important improvements would be to generalise our results to the full non-linear transformation rules and to address the issue of dynamics as well as symmetries.

2.1 Division algebras

- Mathematicians deal with four kinds of numbers, called Divison Algebras.
- The Octonions occupy a privileged position :

Name	Symbol	Imaginary parts
Reals	$\mathbb{R}$	0
Complexes	$\mathbb{C}$	1
Quaternions	$\mathbb{H}$	3
Octonions	$\mathbb{O}$	7

Table : Division Algebras

- They provide an intuitive basis for the exceptional Lie algebras:

Classical algebras		Rank	Dimension
A_n	$SU(n+1)$	n	$(n+1)^2-1$
B_n	$SO(2n+1)$	n	$n(2n+1)$
C_n	$Sp(2n)$	n	$n(2n+1)$
D_n	$SO(2n)$	n	$n(2n-1)$
Exceptional algebras			
E_6		6	78
E_7		7	133
E_8		8	248
F_4		4	52
G_2		2	14

Table : Classical and exceptional Lie alebras

2.3 Magic square

- Freudenthal-Rozenfeld-Tits magic square

$\mathbb{A}_L/\mathbb{A}_R$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	A_1	A_2	C_3	F_4
$\mathbb{C}$	A_2	$A_2 + A_2$	A_5	E_6
$\mathbb{H}$	C_3	A_5	D_6	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

Table : Magic square

- Despite much effort, however, it is fair to say that the ultimate physical significance of octonions and the magic square remains an enigma.

2.4 Octonions

- Octonion x given by
$x = x^0 e_0 + x10 e_1 + x^2 e_2 + x^3 e_3 + x^4 e_4 + x^5 e_5 + x^6 e_6 + x^7 e_7$,
One real $e_0 = 1$ and seven $e_i, i = 1, \ldots, 7$ imaginary elements, where $e_0^* = e_0$ and $e_i^* = -e_i$.
- The imaginary octonionic multiplication rules are,

$$e_i e_j = -\delta_{ij} + C_{ijk} e_k \quad [e_i, e_j, e_k] = 2Q_{ijkl} e_l$$

C_{mnp} are the octonionic structure constants, the set of oriented lines of the Fano plane.

$$\{ijk\} = \{124, 235, 346, 457, 561, 672, 713\}.$$

- Q_{ijkl} are the associators the set of oriented quadrangles in the Fano plane:

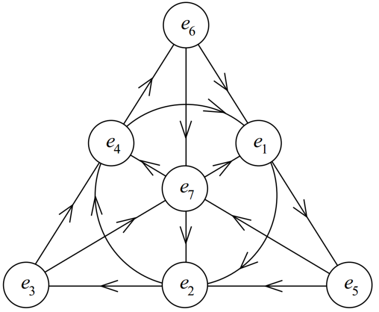
$$ijkl = \{3567, 4671, 5712, 6123, 7234, 1345, 2456\},$$

$$Q_{ijkl} = -\frac{1}{3!} C_{mnp} \varepsilon_{mnpijkl}.$$

2.5 Fano plane

The Fano plane has seven points and seven lines (the circle counts as a line) with three points on every line and three lines through every point.

Fano plane



2.6 Gino Fano

Gino Fano (5 January 1871 to 8 November 1952) was an Italian mathematician. He was born in Mantua and died in Verona.
Fano worked on projective and algebraic geometry; the Fano plane, Fano fibration, Fano surface, and Fano varieties are named for him.
Ugo Fano and Robert Fano were his sons.

2.8 Cayley-Dickson

- Octonion

$$O = O^0 e_0 + O^1 e_1 + O^2 e_2 + O^3 e_3 + O^4 e_4 + O^5 e_5 + O^6 e_6 + O^7 e_7$$
$$= H(0) + e_3 H(1)$$

- Quaternion

$$H(0) = O^0 e_0 + O^1 e_1 + O^2 e_2 + O^4 e_4 \quad H(1) = O^3 e_0 - O^7 e_1 - O^5 e_2 + O^6 e_4$$
$$H(0) = C(00) + e_2 C(10) \quad H(1) = C(01) + e_2 C(11)$$

- Complex

$$C(00) = O^0 e_0 + O^1 e_1 \quad C(01) = O^3 e_0 - O^7 e_1$$
$$C(10) = O^2 e_0 - O^4 e_1 \quad C(11) = -O^5 e_0 - O^6 e_1$$
$$C(00) = R(000) + e_1 R(100) \quad C(01) = R(001) + e_1 R(101)$$
$$C(10) = R(010) + e_1 (110) \quad C(11) = R(011) + e_1 R(111)$$

- Real

$$R(000) = O^0 \;\; R(100) = O^1 \quad R(001) = O^3 \;\; R(101) = -O^7$$
$$R(010) = O^2 \;\; R(110) = -O^4 \quad R(011) = -O^5 \;\; R(111) = -O^6$$

2.7 Division algebras

- Division: ax+b=0 has a unique solution
- Associative: a(bc)=(ab)c
- Commutative: ab=ba

A	construction	division?	associative?	commutative?	ordered?
R	R	yes	yes	yes	yes
C	$R + e_1 R$	yes	yes	yes	no
H	$C + e_2 C$	yes	yes	no	no
O	$H + e_3 H$	yes	no	no	no
S	$O + e_4 O$	no	no	no	no

- As we shall see, the mathematical cut-off of division algebras at octonions corresponds to a physical cutoff at 16 component spinors in super Yang-Mills.

2.8 Norm-preserving algebras

- To understand the symmetries of the magic square and its relation to YM we shall need in particular two algebras defined on $\mathbb{A}$.
- First, the algebra $\mathfrak{norm}(\mathbb{A})$ that preserves the norm

$$\langle x|y\rangle := \frac{1}{2}(x\overline{y} + y\overline{x}) = x^a y^b \delta_{ab}$$

$$\mathfrak{norm}(\mathbb{R}) = 0$$

$$\mathfrak{norm}(\mathbb{C}) = \mathfrak{so}(2)$$

$$\mathfrak{norm}(\mathbb{H}) = \mathfrak{so}(4)$$

$$\mathfrak{norm}(\mathbb{O}) = \mathfrak{so}(8)$$

2.9 Triality Algebra

- Second, the triality algebra $\mathfrak{tri}(\mathbb{A})$

$$\mathfrak{tri}(\mathbb{A}) \equiv \{(A, B, C) | A(xy) = B(x)y + xC(y)\}, \quad A, B, C \in \mathfrak{so}(n), \quad x, y \in \mathbb{A}.$$

$$\begin{aligned} \mathfrak{tri}(\mathbb{R}) &= 0 \\ \mathfrak{tri}(\mathbb{C}) &= \mathfrak{so}(2) + \mathfrak{so}(2) \\ \mathfrak{tri}(\mathbb{H}) &= \mathfrak{so}(3) + \mathfrak{so}(3) + \mathfrak{so}(3) \\ \mathfrak{tri}(\mathbb{O}) &= \mathfrak{so}(8) \end{aligned}$$

[Barton and Sudbery:2003]:

3.1 Yang-Mills spin-off: interesting in its own right

- We give a unified description of
  $D = 3$ Yang-Mills with $\mathcal{N} = 1, 2, 4, 8$
  $D = 4$ Yang-Mills with $\mathcal{N} = 1, 2, 4$
  $D = 6$ Yang-Mills with $\mathcal{N} = 1, 2$
  $D = 10$ Yang-Mills with $\mathcal{N} = 1$
  in terms of a pair of division algebras $(\mathbb{A}_n, \mathbb{A}_{n\mathcal{N}})$, $n = D - 2$
- We present a master Lagrangian, defined over $\mathbb{A}_{n\mathcal{N}}$-valued fields, which encapsulates all cases.
- The overall (spacetime plus internal) on-shell symmetries are given by the corresponding *triality* algebras.
- We use imaginary $\mathbb{A}_{n\mathcal{N}}$-valued auxiliary fields to close the non-maximal supersymmetry algebra off-shell. The failure to close off-shell for maximally supersymmetric theories is attributed directly to the non-associativity of the octonions.
  [arXiv:1309.0546
  A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy]

3.2 $D = 3, \mathcal{N} = 8$ Yang-Mills

- The $D = 3$, $\mathcal{N} = 8$ super YM action is given by

$$S = \int d^3x \left(-\frac{1}{4} F^A_{\mu\nu} F^{A\mu\nu} - \frac{1}{2} D_\mu \phi^A_i D^\mu \phi^A_i + i\bar{\lambda}^A_a \gamma^\mu D_\mu \lambda^A_a \right.$$
$$\left. -\frac{1}{4} g^2 f_{BC}{}^A f_{DE}{}^A \phi^B_i \phi^D_i \phi^C_j \phi^E_j - g f_{BC}{}^A \phi^B_i \bar{\lambda}^{Aa} \Gamma^i_{ab} \lambda^{Cb} \right),$$

  where the Dirac matrices Γ^i_{ab}, $i = 1, \ldots, 7$, $a, b = 0, \ldots, 7$, belong to the SO(7) Clifford algebra.

- The key observation is that Γ^i_{ab} can be represented by the octonionic structure constants,

$$\Gamma^i_{ab} = i(\delta_{bi}\delta_{a0} - \delta_{b0}\delta_{ai} + C_{iab}),$$

  which allows us to rewrite the action over octonionic fields

3.3 Transformation rules

- The supersymmetry transformations in this language are given by

$$
\begin{aligned}
\delta\lambda^A &= \frac{1}{2}(F^{A\mu\nu} + \varepsilon^{\mu\nu\rho} D_\rho \phi^A)\sigma_{\mu\nu}\epsilon + \frac{1}{2} g f_{BC}{}^A \phi_i^B \phi_j^C \sigma_{ij}\epsilon, \\
\delta A_\mu^A &= \frac{i}{2}(\bar{\epsilon}\gamma_\mu\lambda^A - \bar{\lambda}^A\gamma_\mu\epsilon), \\
\delta\phi^A &= \frac{i}{2} e_i[(\bar{\epsilon}e_i)\lambda^A - \bar{\lambda}^A(e_i\epsilon)],
\end{aligned}
\tag{2}
$$

where $\sigma_{\mu\nu}$ are the generators of $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SO}(1,2)$.

4.1 Global symmetries of supergravity in D=3

- MATHEMATICS: Division algebras: R, C, H, O

  $(DIVISION\ ALGEBRAS)^2 = MAGIC\ SQUARE\ OF\ LIE\ ALGEBRAS$

- PHYSICS: $N = 1, 2, 4, 8\ D = 3\ Yang - Mills$

  $(YANG - MILLS)^2 = MAGIC\ SQUARE\ OF\ SUPERGRAVITIES$

- CONNECTION: $N = 1, 2, 4, 8 \sim R, C, H, O$

  $MATHEMATICS\ MAGIC\ SQUARE = PHYSICS\ MAGIC\ SQUARE$

- The $D = 3\ G/H\ grav$ symmetries are given by ym symmetries

$$G(grav) = \mathfrak{tri}\ ym(L) + \mathfrak{tri}\ ym(R) + 3[ym(L) \times ym(R)].$$

eg

$$E_{8(8)} = SO(8) + SO(8) + 3(\mathbb{O} \times \mathbb{O})$$
$$248 = 28 + 28 + (8_v, 8_v) + (8_s, 8_s) + (8_c, 8_c)$$

4.2 Squaring $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ Yang-Mills in $D = 3$

Taking a left SYM multiplet

$$\{A_\mu(L) \in \text{Re}\mathbb{A}_L, \quad \phi(L) \in \text{Im}\mathbb{A}_L, \quad \lambda(L) \in \mathbb{A}_L\}$$

and tensoring with a right multiplet

$$\{A_\mu(R) \in \text{Re}\mathbb{A}_R, \quad \phi(R) \in \text{Im}\mathbb{A}_R, \quad \lambda(R) \in \mathbb{A}_R\}$$

we obtain the field content of a supergravity theory valued in both $\mathbb{A}_L$ and $\mathbb{A}_R$:

$\mathbb{A}_L/\mathbb{A}_R$	$A_\mu(R) \in \text{Re}\mathbb{A}_R$	$\phi(R) \in \text{Im}\mathbb{A}_R$	$\lambda(R) \in \mathbb{A}_R$
$A_\mu(L) \in \text{Re}\mathbb{A}_L$	$g_{\mu\nu} + \varphi \in \text{Re}\mathbb{A}_L \otimes \text{Re}\mathbb{A}_R$	$\varphi \in \text{Re}\mathbb{A}_L \otimes \text{Im}\mathbb{A}_R$	$\Psi_\mu + \chi \in \text{Re}\mathbb{A}_L \otimes \mathbb{A}_R$
$\phi(L) \in \text{Im}\mathbb{A}_L$	$\varphi \in \text{Im}\mathbb{A}_L \otimes \text{Re}\mathbb{A}_R$	$\varphi \in \text{Im}\mathbb{A}_L \otimes \text{Im}\mathbb{A}_R$	$\chi \in \text{Im}\mathbb{A}_L \otimes \mathbb{A}_R$
$\lambda(L) \in \mathbb{A}_L$	$\Psi_\mu + \chi \in \mathbb{A}_L \otimes \text{Re}\mathbb{A}_R$	$\chi \in \mathbb{A}_L \otimes \text{Im}\mathbb{A}_R$	$\varphi \in \mathbb{A}_L \otimes \mathbb{A}_R$

Grouping spacetime fields of the same type we find,

$$g_{\mu\nu} \in \mathbb{R}, \quad \Psi_\mu \in \begin{pmatrix} \mathbb{A}_L \\ \mathbb{A}_R \end{pmatrix}, \quad \varphi \in \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}, \quad \chi \in \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}$$

4.3 Grouping together

- Grouping spacetime fields of the same type we find,

$$g_{\mu\nu} \in \mathbb{R}, \quad \Psi_\mu \in \begin{pmatrix} \mathbb{A}_L \\ \mathbb{A}_R \end{pmatrix}, \quad \varphi, \chi \in \begin{pmatrix} \mathbb{A}_L \otimes \mathbb{A}_R \\ \mathbb{A}_L \otimes \mathbb{A}_R \end{pmatrix}. \tag{3}$$

- Note we have dualised all resulting p-forms, in particular vectors to scalars. The $\mathbb{R}$-valued graviton and $\mathbb{A}_L \oplus \mathbb{A}_R$-valued gravitino carry no degrees of freedom. The $(\mathbb{A}_L \otimes \mathbb{A}_R)^2$-valued scalar and Majorana spinor each have $2(\dim \mathbb{A}_L \times \dim \mathbb{A}_R)$ degrees of freedom.
- Fortunately, $\mathbb{A}_L \oplus \mathbb{A}_R$ and $(\mathbb{A}_L \otimes \mathbb{A}_R)^2$ are precisely the representation spaces of the vector and (conjugate) spinor. For example, in the maximal case of $\mathbb{A}_L, \mathbb{A}_R = \mathbb{O}$, we have the $\mathbf{16}, \mathbf{128}$ and $\mathbf{128}'$ of SO(16).

4.4 Final result

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathcal{N}=2, f=4$ $G=\mathrm{SL}(2,\mathbb{R})$, dim 3 $H=\mathrm{SO}(2)$, dim 1	$\mathcal{N}=3, f=8$ $G=\mathrm{SU}(2,1)$, dim 8 $H=\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 4	$\mathcal{N}=5, f=16$ $G=\mathrm{USp}(4,2)$, dim 21 $H=\mathrm{USp}(4)\times\mathrm{USp}(2)$, dim 13	$\mathcal{N}=9, f=32$ $G=F_{4(-20)}$, dim 52 $H=\mathrm{SO}(9)$, dim 36
$\mathbb{C}$	$\mathcal{N}=3, f=8$ $G=\mathrm{SU}(2,1)$, dim 8 $H=\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 4	$\mathcal{N}=4, f=16$ $G=\mathrm{SU}(2,1)^2$, dim 16 $H=\mathrm{SU}(2)^2\times\mathrm{SO}(2)^2$, dim 8	$\mathcal{N}=6, f=32$ $G=\mathrm{SU}(4,2)$, dim 35 $H=\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 19	$\mathcal{N}=10, f=64$ $G=E_{6(-14)}$, dim 78 $H=\mathrm{SO}(10)\times\mathrm{SO}(2)$, dim 46
$\mathbb{H}$	$\mathcal{N}=5, f=16$ $G=\mathrm{USp}(4,2)$, dim 21 $H=\mathrm{USp}(4)\times\mathrm{USp}(2)$, dim 13	$\mathcal{N}=6, f=32$ $G=\mathrm{SU}(4,2)$, dim 35 $H=\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 19	$\mathcal{N}=8, f=64$ $G=\mathrm{SO}(8,4)$, dim 66 $H=\mathrm{SO}(8)\times\mathrm{SO}(4)$, dim 34	$\mathcal{N}=12, f=128$ $G=E_{7(-5)}$, dim 133 $H=\mathrm{SO}(12)\times\mathrm{SO}(3)$, dim 69
$\mathbb{O}$	$\mathcal{N}=9, f=32$ $G=F_{4(-20)}$, dim 52 $H=\mathrm{SO}(9)$, dim 36	$\mathcal{N}=10, f=64$ $G=E_{6(-14)}$, dim 78 $H=\mathrm{SO}(10)\times\mathrm{SO}(2)$, dim 46	$\mathcal{N}=12, f=128$ $G=E_{7(-5)}$, dim 133 $H=\mathrm{SO}(12)\times\mathrm{SO}(3)$, dim 69	$\mathcal{N}=16, f=256$ $G=E_{8(8)}$, dim 248 $H=\mathrm{SO}(16)$, dim 120

- The $\mathcal{N}>8$ supergravities in $D=3$ are unique, all fields belonging to the gravity multiplet, while those with $\mathcal{N}\leq 8$ may be coupled to k additional matter multiplets [Marcus and Schwarz:1983; deWit, Tollsten and Nicolai:1992]. The real miracle is that tensoring left and right YM multiplets yields the field content of $\mathcal{N}=2,3,4,5,6,8$ supergravity with $k=1,1,2,1,2,4$: just the right matter content to produce the U-duality groups appearing in the magic square.

4.5. Conclusion

- In both cases the field content is such that the U-dualities exactly match the groups of of the magic square:

$\mathbb{A}_L/\mathbb{A}_R$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$SL(2,\mathbb{R})$	$SU(2,1)$	$USp(4,2)$	$F_{4(-20)}$
$\mathbb{C}$	$SU(2,1)$	$SU(2,1)\times SU(2,1)$	$SU(4,2)$	$E_{6(-14)}$
$\mathbb{H}$	$USp(4,2)$	$SU(4,2)$	$SO(8,4)$	$E_{7(-5)}$
$\mathbb{O}$	$F_{4(-20)}$	$E_{6(-14)}$	$E_{7(-5)}$	$E_{8(8)}$

Table : Magic square

- The G/H U-duality groups are precisely those of the Freudenthal Magic Square!

$$G: \quad \mathfrak{g}_3(\mathbb{A}_L,\mathbb{A}_R) := \mathfrak{tri}(\mathbb{A}_L)+\mathfrak{tri}(\mathbb{A}_R)+3(\mathbb{A}_L\times\mathbb{A}_R).$$

$$H: \quad \mathfrak{g}_1(\mathbb{A}_L,\mathbb{A}_R) := \mathfrak{tri}(\mathbb{A}_L)+\mathfrak{tri}(\mathbb{A}_R)+(\mathbb{A}_L\times\mathbb{A}_R).$$

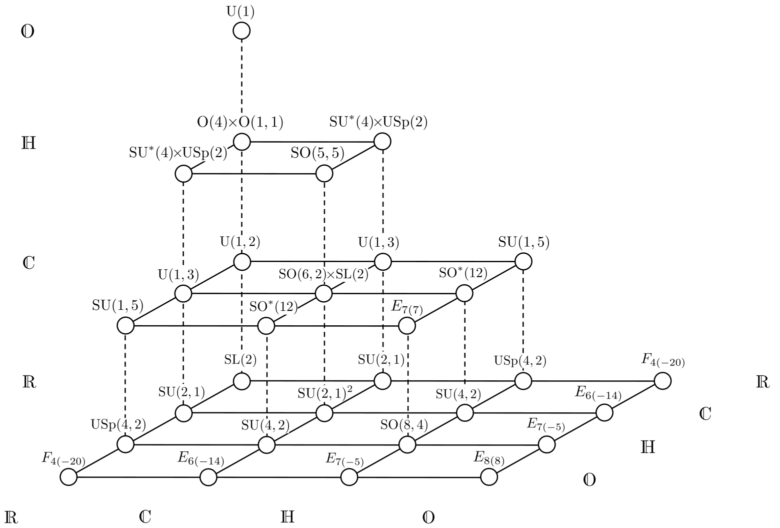
4.5a Projective planes?

- U-dualities G are realised non-linearly on the scalars, which parametrise the symmetric spaces G/H.
- This can be understood using the remarkable identity relating the projective planes over $(\mathbb{A}_L \otimes \mathbb{A}_R)^2$ to G/H,

$$(\mathbb{A}_L \otimes \mathbb{A}_R)\mathbb{P}^2 \cong G/H.$$

  The scalar fields may be regarded as points in division-algebraic projective planes [Baez:2001, Freudenthal:1964, Landsberg2001].
- See also [Atiyah and Berndt:2002].

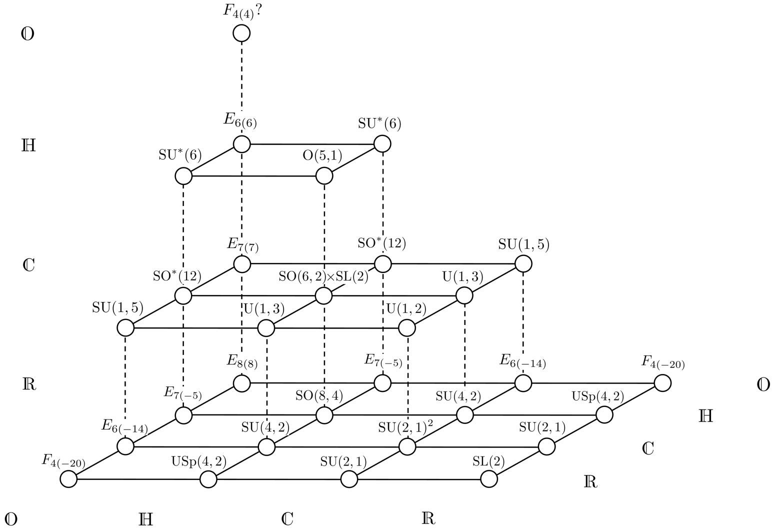
4.6 Magic Pyramid: G symmetries



4.7 Summary Gravity: Conformal Magic Pyramid

- We also construct a *conformal* magic pyramid by tensoring conformal supermultiplets in D = 3, 4, 6.
- The missing entry in D = 10 is suggestive of an exotic theory with G/H duality structure $F_{4(4)}/Sp(3) \times Sp(1)$.

4.8 Conformal Magic Pyramid: G symmetries



5.1 Generalized mirror symmetry: M-theory on X^7

- We consider compactification of $(\mathcal{N} = 1, D = 11)$ supergravity on a 7-manifold X^7 with betti numbers $(b_0, b_1, b_2, b_3, b_3, b_2, b_1, b_0)$ and define a generalized mirror symmetry

$$(b_0, b_1, b_2, b_3) \to (b_0, b_1, b_2 - \rho/2, b_3 + \rho/2)$$

under which

$$\rho(X^7) \equiv 7b_0 - 5b_1 + 3b_2 - b_3$$

changes sign

$$\rho \to -\rho$$

[Duff and Ferrara:2010]

- Generalized self-mirror theories are defined to be those for which $\rho = 0$

	Field	f	ΔA	$360A$	$360A'$	X^7
g_{MN}	$g_{\mu\nu}$	2	-3	848	-232	b_0
	$\mathcal{A}_\mu$	2	0	-52	-52	b_1
	$\mathcal{A}$	1	0	4	4	$-b_1 + b_3$
ψ_M	ψ_μ	2	1	-233	127	$b_0 + b_1$
	χ	2	0	7	7	$b_2 + b_3$
A_{MNP}	$A_{\mu\nu\rho}$	0	2	-720	0	b_0
	$A_{\mu\nu}$	1	-1	364	4	b_1
	A_μ	2	0	-52	-52	b_2
	A	1	0	4	4	b_3
total ΔA			0			
total A				$-\rho/24$		
total A'					$-\rho/24$	

Table : X^7 compactification of D=11 supergravity

5.3 Vanish without a trace!

- Remarkably, we find that the anomalous trace depends on ρ

$$A = -\frac{1}{24}\rho(X^7)$$

  So the anomaly flips sign under generalized mirror symmetry and vanishes for generalized self-mirror theories.

- Equally remarkable is that we get the same answer for the total trace using the numbers of Grisaru et al 1980.

5.4 Squaring Yang-Mills in $D = 4$ and the self-mirror condition

- Tensoring left and right supersymmetric Yang-Mills theories with field content $(A_\mu, N_L\chi, 2(N_L-1)\phi)$ and $(A_\mu, N_R\chi, 2(N_R-1)\phi)$ yields an $N = N_L + N_R$ supergravity theory.

$L \setminus R$	A_μ	$N_R\chi$	$2(N_R-1)\phi$
A_μ	$g_{\mu\nu} + 2\phi$	$N_R(\psi_\mu + \chi)$	$2(N_R-1)A_\mu$
$N_L\chi$	$N_L(\psi_\mu + \chi)$	$N_L N_R(A_\mu + 2\phi)$	$2N_L(N_R-1)\chi$
$2(N_L-1)\phi$	$2(N_L-1)A_\mu$	$2(N_L-1)N_R\chi$	$4(N_L-1)(N_R-1)\phi$

Table : Tensoring N_L and N_R super Yang-Mills theories in $D = 4$. Note that we have dualized the 2-form coming from the vector-vector slot

5.5 Betti numbers from squaring Yang-Mills

- The betti numbers may then be read off from the Table and we find

$$(b_0, b_1, b_2, b_3) =$$

$$(1, N_L + N_R - 1, N_L N_R + N_L + N_R - 3, 3N_L N_R - 2N_L - 2N_R + 3)$$

Consequently

$$\rho\left(X^7\right) = 7b_0 - 5b_1 + 3b_2 - b_3 = 0 \tag{4}$$

- Similar results hold in $D = 5$ where

$$(c_0, c_1, c_2, c_3) = (1, N_L + N_R - 2, N_L N_R - 1, 2N_L N_R - 2N_L - 2N_R + 4)$$

Consequently

$$\chi\left(X^6\right) = 2b_0 - 2c_1 + 2c_2 - c_3 = 0 \tag{5}$$