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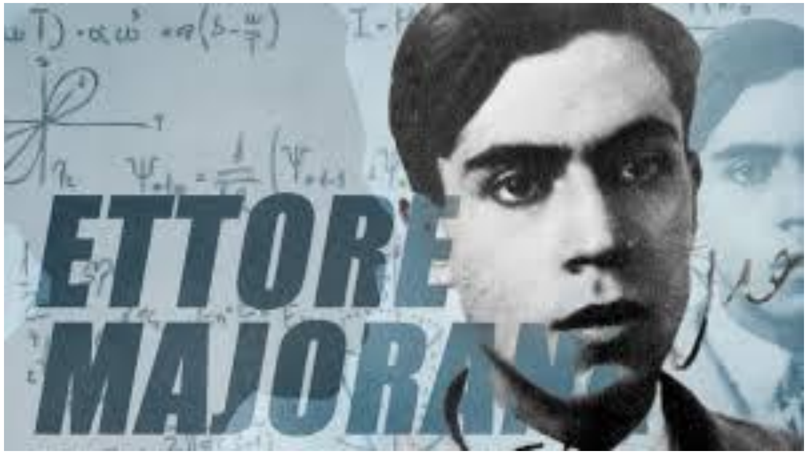

Majorana Fermions in Particle Physics, Solid State and Quantum Information Theory

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25 June 2015, EMFCSC Erice

Majorana Fermions

- Ettore Majorana (1906-1938) disappeared while traveling by ship from Palermo to Naples in 1938. His demise intrigues us still today because of his seminal work, published the previous year (cajoled by Fermi), that established solutions to the Dirac equation that describe a fermionic particle that is its own anti-particle.

- Majorana, E. Nuovo Cimento 5, 171 –184 (1937).

Dirac's equation connects the four components of a field ψ:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0,$$

where

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$$

Majorana found a representation such that $i\gamma^\mu$ are purely real:

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & +i & 0 & 0 \\ +i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & +i & 0 \\ 0 & 0 & 0 & +i \\ +i & 0 & 0 & 0 \\ 0 & +i & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} +i & 0 & 0 & 0 \\ 0 & +i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \\ 0 & +i & 0 & 0 \\ +i & 0 & 0 & 0 \end{pmatrix}.$$

Hence the Dirac equation can describe a real field ψ.

Majorana’s Influence

- Particle Physics
- Solid State Physics
- Quantum Information Theory

Particle Physics

- Neutrinos: see-saw, beta decay
- Gell-Mann, M., Ramond, P. & Slansky, R. in Supergravity (eds Freedman, D. & van Nieuwenhuizen, P.) 315 North Holland, 1979.
- http://en.wikipedia.org/wiki/Double beta decay
- Supersymmetry : MSSM, supergravity, string/M theory
- (see Ferrara's talk)

Solid State Physics

- Majorana fermions in superconductors
- Majorana zero-modes (MZMs) in D=1 condensed matter systems
- ``Anyons in an exactly solved model and beyond'', Kitaev, cond-mat/0506438
- ``Observation of Majorana Fermions in Ferromagnetic Atomic Chains on a Superconductor'' Nadj-Perge, Drozdov, Hua Chen, Jeon, MacDonald, Bernevig &Yazdani, arXiv:1410.0682
- ``A chain of iron atoms on lead may reveal a signature of the elusive Majorana particle'', Lee, Science, June 2015

Quantum Information Theory

- MZMs might lead to fault-tolerant quantum computers
- ``Fermionic Quantum Computation'', Bravy & Kitaev
- ``Non-Abelian anyons and topological quantum computation'', Nayek et al, Rev. Mod. Phys., Vol. 80, 2008.
- ``Fault-tolerant quantum computers'', Preskill, 2005, quant-ph/9712048
- Superqubits
- Borsten, Dahanayake, Duff & Rubens, “Superqubits,” arXiv:0908.0706
- Borsten, Bradler & Duff, “Tsirelson’s bound and supersymmetric entangled states,” arXiv:1206.6934
- Castellani, Grassi & Sommovigo, arXiv:0904.2512
- Borsten, Bradler and Duff, ``**Generating entangled superqubit states**“ arXiv:1411.7311
-

Warning

- Much of the recent excitement in condensed matter physics is about the theoretical utility and possible experimental observation of ``Majorana zero modes''.
- Excitement is justified; may lead to breakthrough in quantum computing.
- BUT…

Misnomer

- The term “Majorana” refers to the fact that the relevant operators are real, as in Majorana’s real version of the Dirac equation. However, there is little connection with Majorana’s original work or its application to neutrinos. Rather, the key concept here is the non-Abelian anyon, and MZMs are a particular mechanism by which a particular type of non-Abelian anyons, usually called “Ising anyons” can arise.

- By contrast, Majorana fermions, as originally conceived, obey ordinary Fermi-Dirac statistics, and are simply a particular type of fermion. Although the terminology ‘Majorana fermions’ is is used extensively in the literature for MZMs, it is somewhat misleading.

``Majorana Zero Modes and Topological Quantum Computation''

Das Sarma, Freedman and Nayak, cond-mat 1501.02813

Reviews

- ``Majorana returns'', Wilczek *Nature Physics* **5**, 614 - 618 (2009)

- Colloquium: Majorana Fermions in ``Nuclear, Particle and Solid-state Physics'' Elliott & Franzy, cond-mat 1403.4976

- ``New directions in the pursuit of Majorana fermions in solid state systems'' Alicea, cond-mat 1202.1293

- ``A solid case for Majorana fermions'', Reich, Nature, 06 March 2012 Updated: 13 April 2012.

PARTICLE PHYSICS

Particle = antiparticle

Zero charge necessary eg

Photon

Graviton

Pi-zero

But not sufficient eg

Neutron

So do we need/want a spin ½ partlcle=antiparticle?

Neutrinos

Particles emitted in $\pi^+ \to \mu^+ + \nu_\mu$ will induce neutron-to-proton conversion $\nu_\mu + n \to \mu^- + p$ but not proton-to-neutron. Such reactions favour separate L_e, L_μ and L_τ lepton-number conservation with ν and $\bar{\nu}$ having opposite lepton number.

BUT

Neutrinos can oscillate between flavors: a ν_e emitted from the sun can arrive on earth as a ν_μ or ν_τ so maybe only $L_e + L_\mu + L_\tau$ is conserved.

Maybe neutrinos are Majorana after all? Test would be neutrinoless double beta decay e.g. $Ge^{76} \to Se^{76} + 2e$

SOLID STATE PHYSICS

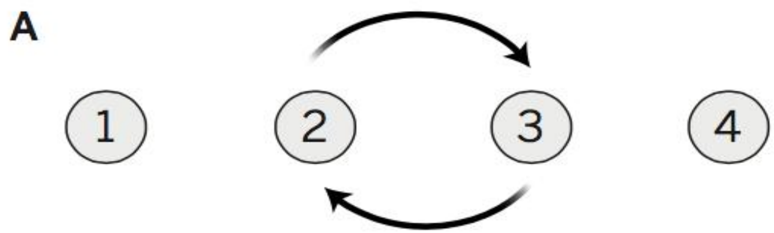
MBS

- First, the quasi-particles in a superconductor are natural candidates for Majorana physics, because they constitute a quantum mechanical admixture of particles and holes. If the admixture has equal amplitude, the antiparticles and particles become identical.

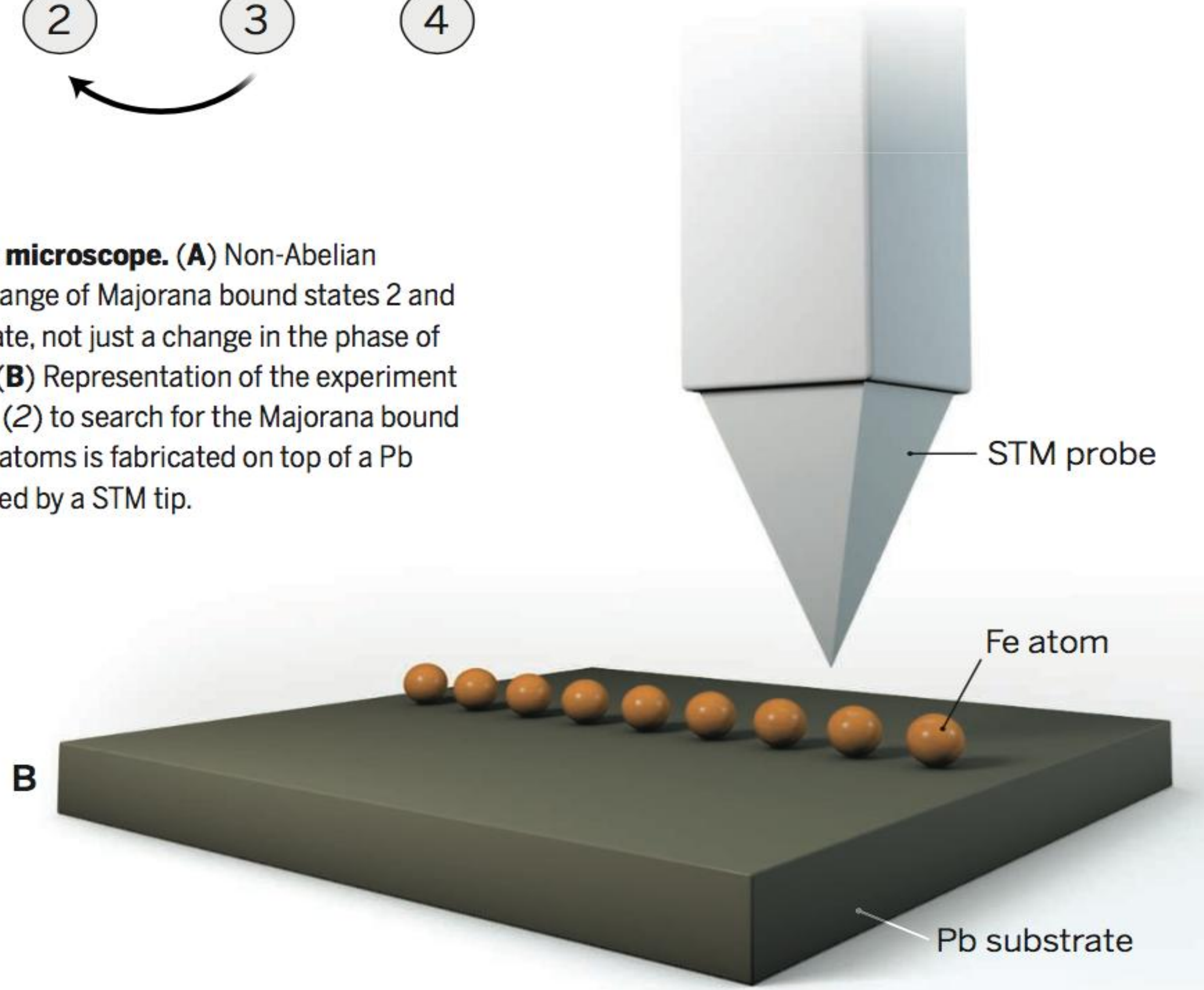
- Kitaev : Majorana bound states (MBSs) form at the ends of a superconductor chain if the wave function of the electron pair formed in the superconductor is antisymmetric (called p-wave), as opposed to symmetric (s-wave) in conventional superconductors.

Quantum Computing

- The MBSs obey what is referred to as non-Abelian statistics—an exotic possibility that has not been observed experimentally.
- The ability to spatially decompose quasi-particle states has been proposed to be the basis of a fault-tolerant quantum computer and memory.



Exotica under the microscope. (**A**) Non-Abelian statistics: The exchange of Majorana bound states 2 and 3 leads to a new state, not just a change in the phase of the wave function. (**B**) Representation of the experiment of Nadj-Perge *et al*. (*2*) to search for the Majorana bound state. A chain of Fe atoms is fabricated on top of a Pb substrate and probed by a STM tip.



(Kitaev '03)

The degenerate states associated with Majorana zero modes define a topologically protected quantum memory

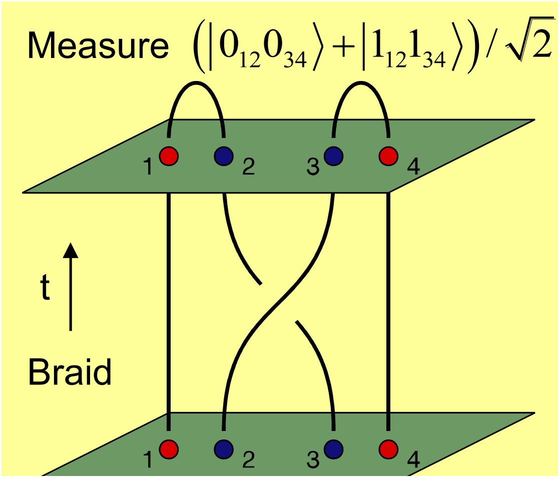
- 2 Majorana separated bound states = 1 fermion $\Psi = \gamma_1 + i\gamma_2$
  - 2 degenerate states (full/empty) = 1 qubit
- 2N separated Majoranas = N qubits
- Quantum Information is stored non locally
  - Immune from local decoherence

Braiding performs unitary operations

Non-Abelian statistics

Interchange rule (Ivanov 03)

$$\gamma_i \to \gamma_j$$
$$\gamma_j \to -\gamma_i$$



QUANTUM INFORMATION THEORY

One qubit

- A qubit is any two-state system
- The one qubit system Alice (where $A = 0, 1$) is described by the state

$$\begin{aligned} |\Psi\rangle &= a_A|A\rangle \\ &= a_0|0\rangle + a_1|1\rangle. \end{aligned}$$

Two qubits

- The two qubit system (Alice and Bob) is described by the state
$$|\Psi\rangle = a_{AB}|AB\rangle$$
$$|\Psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$
- Bipartite entanglement given 2-tangle
$$\tau_{AB} = 4|\det \rho_A| = 4|\det a_{AB}|^2,$$
$$\rho_A = Tr_B|\Psi\rangle\langle\Psi|$$
- Invariant under permutations and
$$SL(2)_A \times SL(2)_B$$

- Example, separable state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle$$

$$\tau_{AB} = 0$$

- Example, Bell state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$\tau_{AB} = 1$$

- EPR “paradox”

Three qubits

- The three qubit system (Alice, Bob, Charlie) is described by the state

$$\begin{aligned} |\Psi\rangle = \;& a_{000}|000\rangle + a_{001}|001\rangle + a_{010}|010\rangle + a_{011}|011\rangle \\ +\;& a_{100}|100\rangle + a_{101}|101\rangle + a_{110}|110\rangle + a_{111}|111\rangle, \end{aligned}$$

- Transforms as a (2, 2, 2) under the SLOCC group

$$SL(2)_A \times SL(2)_B \times SL(2)_C$$

- The tripartite entanglement of Alice, Bob and Charlie is given by the "3-tangle"

$$\tau_{ABC} = 4|\mathrm{Det}\ a_{ABC}|,$$

Coffman et al: quant-ph/9907047

- Det a_{ABC} is Cayley's hyperdeterminant

$$\mathrm{Det}\ a_{ABC} = -\frac{1}{2}\,\varepsilon^{A_1A_2}\varepsilon^{B_1B_2}\varepsilon^{C_1C_4}\varepsilon^{C_2C_3}\varepsilon^{A_3A_4}\varepsilon^{B_3B_4} \cdot a_{A_1B_1C_1}a_{A_2B_2C_2}a_{A_3B_3C_3}a_{A_4B_4C_4}$$

Miyake and Wadati: quant-ph/0212146

Superqubits

- 1845: hyperdeterminant introduced
  Cayley: $[SL(2)]^3$
- 2002: describes 3-qubit entanglement
  Miyake and Wadati:$[SL(2, C)]^3$
- 2009: superhyperdeterminant introduced
  Castellani et al : $[OSp(2|1)]^3$
  Obviously the answer. What is the question?
- 2009: describes 3-superqubit entanglement
  Borsten, Dahanayake, Duff and Rubens

- Stochastic Local Operations and Classical Communication=SLOCC
- Two states are said to be LOCC equivalent if they may b transformed into one another with *certainty* using Local Operations and Classical Communication, and SLOCC with some *non-vanishing probability* (Bennett:1999, Dur:2000).
- For n qubits the LOCC equivalence group is given by $[SU(2)]^n$ and the SLOCC equivalence group by $[SL(2, C)]^n$, one factor for each qubit.

Qubits and Superqubits

- Qubits belong to a 2-dimensional complex Hilbert space:

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle$$

- Superqubits belong to a (2|1)-dimensional complex super-Hilbert space:

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle + a_\bullet|\bullet\rangle$$

- In this sense they constitute the minimal supersymmetric generalisation of the conventional qubit.

- The one superqubit system (Alice) is described by the state

$$|\Psi\rangle = |X\rangle a_X = |A\rangle a_A + |\bullet\rangle a_\bullet,$$

  where a_A is commuting with $A = 0, 1$ and $a_\bullet$ is anticommuting. That is to say, the state vector is promoted to a supervector. The super Hilbert space has dimension 3, two "bosons" and one "fermion".
- The super SLOCC equivalence group for a single qubit is $OSp(2|1)_A$. Under the $SL(2)_A$ subgroup a_A transforms as a **2** while $a_\bullet$ is a singlet.

- The two superqubit system (Alice and Bob) is described by the state

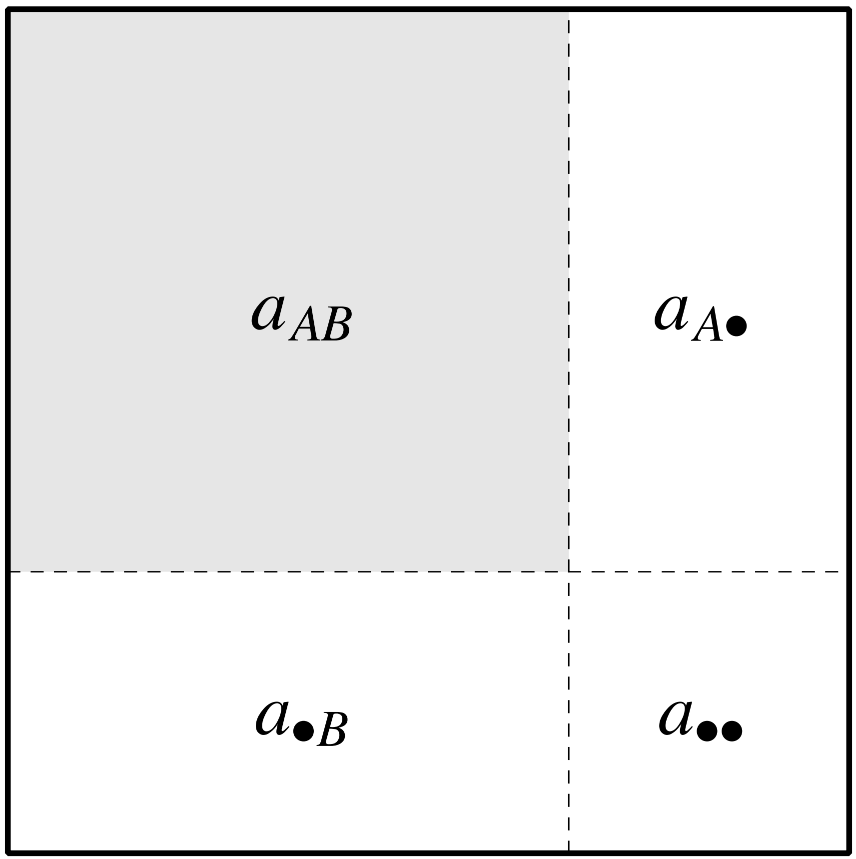
$$|\Psi\rangle = |AB\rangle a_{AB} + |A\bullet\rangle a_{A\bullet} + |\bullet B\rangle a_{\bullet B} + |\bullet\bullet\rangle a_{\bullet\bullet}$$

  where a_{AB} is commuting, $a_{A\bullet}$ and $a_{\bullet B}$ are anticommuting and $a_{\bullet\bullet}$ is commuting. The super Hilbert space has dimension 9: 5 “bosons” and 4 “fermions”.

- The super SLOCC group for two superqubits is $OSp(2|1)_A \times OSp(2|1)_B$. Under the $SL(2)_A \times SL(2)_B$ subgroup a_{AB} transforms as a $(\mathbf{2}, \mathbf{2})$, $a_{A\bullet}$ as a $(\mathbf{2}, \mathbf{1})$, $a_{\bullet B}$ as a $(\mathbf{1}, \mathbf{2})$ and $a_{\bullet\bullet}$ as a $(\mathbf{1}, \mathbf{1})$ as summarised in the following.

$(2|1) \times (2|1)$ supermatrix

$$\langle XY|\Psi\rangle = a_{XY} = \left(\begin{array}{c|c} a_{AB} & a_{A\bullet} \\ \hline a_{\bullet B} & a_{\bullet\bullet} \end{array}\right)$$

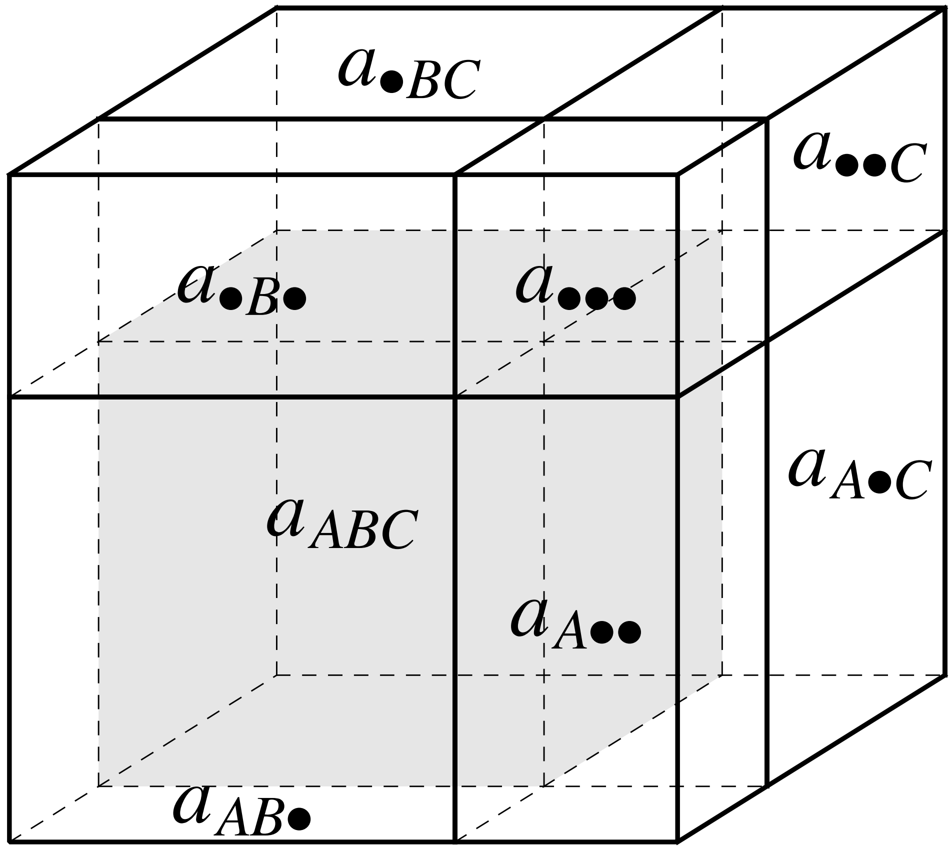


- The three superqubit system (Alice, Bob and Charlie) is described by the state

$$
\begin{aligned}
|\Psi\rangle = {} & |ABC\rangle a_{ABC} \\
& + |AB\bullet\rangle a_{AB\bullet} + |A \bullet C\rangle a_{A\bullet C} + |\bullet BC\rangle a_{\bullet BC} \\
& + |A \bullet \bullet\rangle a_{A\bullet\bullet} + |\bullet B\bullet\rangle a_{\bullet B\bullet} + |\bullet \bullet C\rangle a_{\bullet\bullet C} \\
& + |\bullet \bullet \bullet\rangle a_{\bullet\bullet\bullet}
\end{aligned}
$$

  where a_{AB} is commuting, $a_{AB\bullet}$ $a_{A\bullet C}$ $a_{\bullet BC}$ are anticommuting, $a_{A\bullet\bullet}$ $a_{\bullet B\bullet}$ $a_{\bullet\bullet C}$ are commuting and $a_{\bullet\bullet\bullet}$ is anticommuting.

$(2|1) \times (2|1) \times (2|1)$ superhypermatrix



Are superqubits more nonlocal than qubits?

- A measure of nonlocality is provided by the probability of winning the CHSH game:
- CLASSICAL $p=0.75$
- QUANTUM $p=0.85$
- SUPERQUANTUM $p=0.92$
- NON-SIGNALING $p=1$

WITTEN

Edward Witten, Lecture At Strings 2015 ICTS-TITR, Bangalore, June 22

The example that I want to give is actually very fashionable in current work in condensed matter physics, where it appears in the theory of a (mostly hypothetical) topological superconductor. The theory we consider is simply a two-component massless Majorana fermion in three dimensions, coupled to gravity only

$$I = \int \mathrm{d}^3 x \; i\bar{\psi}\not{D}\psi.$$

If we consider this theory on an oriented three-manifold, then its partition function Z_M is real (because the Dirac operator is hermitian) but not necessarily positive.

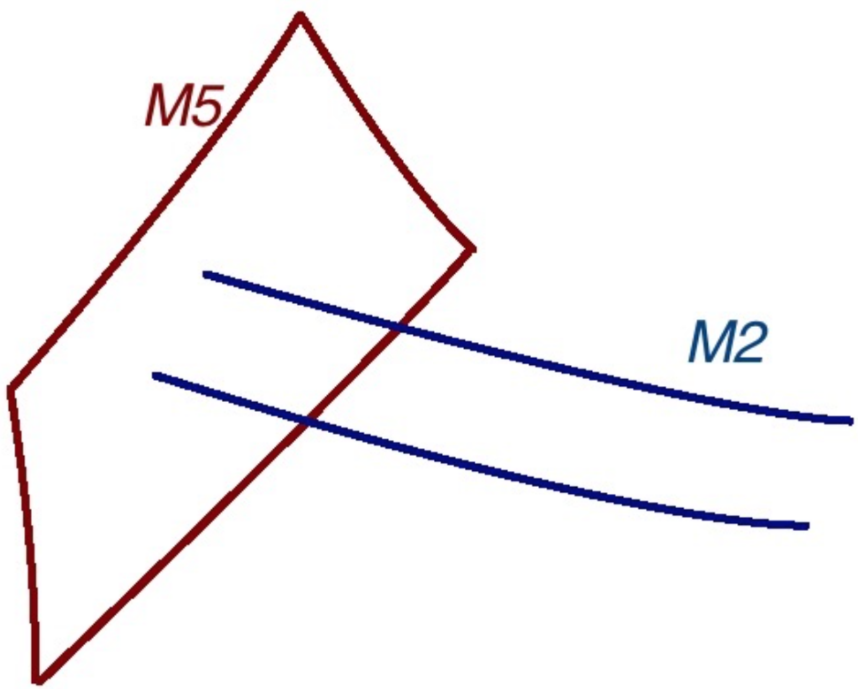
Majorana on unorientable spaces

- I believe this means that the theory is inconsistent even if formulated on orientable manifolds only.
- There is no natural way to choose the sign of $Z\psi(M)$, even though it has no anomaly in the traditional sense.
- It is more interesting, however, to take advantage of the fact that the theory of the massless Majorana fermion is parity-conserving and to try to formulate it on a possibly unorientable three-manifold.

Connection to condensed matter

- One can ask more generally (and many condensed matter physicists have asked this) whether a theory of ν massless Majorana fermions, all transforming the same way under parity, is consistent (on a possibly unorientable three-manifold).
- It turns out that this is so if and only if ν is a multiple of 16.
- This number 16 has been discovered and explained from several different points of view in the condensed matter literature, initially by Kitaev.
- The question as posed by condensed matter physicists is this: For what values of ν is it possible, by adding interactions, to make the boundary fermions gapped while preserving reflection symmetry?

What got me into this subject was thinking about a more subtle case that has not been treated in the literature even at the level of anomalies only: The M2-brane path integral for the case of an M2-brane that ends on an M5-brane.



M-theory

- One can ask what would be a condensed matter analog of the M2-M5 problem.
- A partial analog would be a topological superconductor with 3 + 1-dimensional worldvolume Y, whose boundary M is divided in two parts with two different boundary conditions (possibly because half of the boundary is in contact with some other material).

SUMMARY

Majorana's influence doesn't stop with particle physics, even though that was his original consideration.

- The equations he derived also arise in solid state physics where they describe electronic states in materials with superconducting order.
- In particular, ``Majorana zero modes'' are endowed with some remarkable physica properties that may lead to advances in quantum computing and, in fact, there is evidence that they have been experimentally observed.
- Also permits ''superqubits'' (applications yet unclear)
- Finally, extrapolating Witten: Is M-theory the answer to quantum computing?

Enrico Fermi quote

“There are many categories of scientists: people of second and third rank, who do their best, but do not go very far; there are also people of first-class rank, who make great discoveries, fundamental to the development of science. But then there are the geniuses, like Galileo and Newton. Well Ettore Majorana was one of them.”

Consistency Check

- At first sight this appears to present a conundrum.
- Consistency: bosonic subgroup of $UOSp((3n + 1)/2|(3n - 1)/2)$ is required to act transitively on the subspace of regular qubits sitting inside the super-Hilbert space. However, for $4 > n > 1$ the standard group of global unitaries $SU(2n)$ is not contained in bosonic subgroup of $UOSp((3n + 1)/2|(3n - 1)/2)$.
- Its absence, it would seem, obstructs the expected consistent reduction to standard qubits. All is not lost however, since a proper subgroup $USp(2n) \subset SU(2n)$ is sufficient to generate any state from any initial state.
- This smaller group is indeed always contained in the bosonic subgroup of $UOSp((3n + 1)/2|(3n - 1)/2)$ and, as we shall explain, acts transitively on the subspace of standard n-qubit states.

Generating Entangled States

- Recall, a collection of n distinct isolated qubits transforms under the group of local unitaries [SU(2)]n. The local unitary transformations, by construction, cannot generate entanglement; a separable n-qubit state will remain separable under all [SU(2)]^n operations.

- To generate an arbitrary state, entangled or otherwise, from any given initial state one conventionally employs the group of global unitaries SU(2n), which acts transitively on the n-qubit state space.

Generating Superqubit States

- Similarly, a collection of n distinct isolated superqubits transforms under the local unitary orthosymplectic supergroup [UOSp(2|1)]n, which contains as its bosonic subgroup the conventional local unitaries [SU(2)]n. Again, being local, this set of transformations is insufficient to generate super-entanglement.
- With this issue in mind we introduce here the n-superqubit global unitary supergroup given by UOSp((3n + 1)/2|(3n − 1)/2), the super-analog of SU(2n), which is uniquely determined by the single superqubit limit.